The Method of Images

The uniqueness theorem of the solutions to Poisson's Eqn for given boundary condition is a powerful result that gives rise to creative approaches. It implies that if we can "guess" a solution by any approach that satisfies the boundary conditions, it must be the solution.

A beautiful example is the "method of images", used to find the solution for charges brought close to the surface of conductors. It is not really a "method"; it's more of an "art", involving some good intuition and creativity.

The "classic image problem"

A charge \( q \) is brought a distance \( d \) from an infinite ideal conducting plane. The \( \infty \) conductor is grounded. Find the electric field everywhere.

\[
\begin{array}{c}
\text{\( q \)} \\
\hline \\
\text{\( \frac{1}{2} \)} \\
\end{array}
\]

Physically, we know that oppositely charged surface charge density will be attracted to \( q \), brought in from ground. But how is this charge distributed? And how to we find \( E \) from that charge configuration?
What are the boundary conditions?

\( V = 0 \)

\( \sigma \)

\( z \)

\( s, \; V = 0 \)

1. \( V = 0 \) on plane \( z = 0 \)
2. \( V = 0 \) at \( \infty \)
3. \( \text{Charge } q \) at \( z = d \)

We break up the ambient space into two regions, \( z > 0 \) and \( z < 0 \). For \( z < 0 \), \( V \) satisfies Laplace's equation at \( V = 0 \) on the surrounding boundary (plane + hemisphere \( \rightarrow \infty \))

\( \Rightarrow \text{ } V = 0 \) everywhere for \( z < 0 \)

\( \Rightarrow \text{ For } z > 0 \) \( V \) is a solution with charge \( q \) at \( z = d \) and \( V = 0 \) at \( z = 0 \) and \( \text{in} \rightarrow \infty \)

Trick: In finding the solution for \( z > 0 \), we can "mock up" the field generated by the surfaces charges on the plane by any charge distribution placed in the region \( z < 0 \). If this distribution plus \( q \) produces a potential which satisfies all the boundary conditions then it is the solution for \( z > 0 \).

Note: This does not give the solution for \( z < 0 \).
"Dual" Problem: Find a charge distribution in $z < 0$ which together with $q_0 @ z = 0$ make $V = 0$ on conducting plane.

Guess: An "image" charge of equal and opposite magnitude @ $z = -d$

\[
V(x, y, z=0) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2}} - \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2}} = 0
\]

at $V \Rightarrow 0 \Rightarrow |x| \gg \infty$

$\Rightarrow$ The image charge @ $z = -d$ makes up the surface charges to produce the same field for $z > 0$

Image Problem

True problem
The Solution is thus

\[ V(x, y, z) = \begin{cases} \frac{q}{4\pi \varepsilon_0 \sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{4\pi \varepsilon_0 \sqrt{x^2 + y^2 + (z+d)^2}} & \text{for } z > 0 \\ 0 & \text{for } z < 0 \end{cases} \]

Given \( V(r) \), we can find the true induced surface charge density based on the discontinuity relations of \( \vec{E} \).

Recall: \( \Delta \vec{E}_z = \frac{\partial V}{\partial z} \)

\[ \vec{E}_z = \begin{cases} \frac{\partial V}{\partial z} & \text{for } z > 0 \\ 0 & \text{for } z < 0 \end{cases} \]

\[ \Rightarrow \quad \sigma(x, y) = \frac{\partial V}{\partial z} \bigg|_{z=0} = \frac{q}{4\pi} \sum_{j=1}^{3} \frac{-(z - d)}{((x_j^2 + y_j^2) + (z - d)^2)^{3/2}} \]

\[ \Rightarrow \quad \sigma(x, y) = \frac{-qd}{2\pi (x^2 + y^2 + d^2)^{3/2}} \quad \text{Axial} \]

\[ \text{Symmetric} \]

Total induced charge

\[ q_{\text{induced}} = \int_{\text{plane}} d\sigma = \int_0^\infty 2\pi rdr \sigma(r) \]

\[ = -qd \int_0^\infty \frac{rdr}{(r^2 + d^2)^{3/2}} \]

\[ = -qd \left[ \frac{-1}{(r^2 + d^2)^{1/2}} \right]_0^\infty = -q \]
Induced force

Because of the induced surface charge, the point charge $q$ will be attracted to the plane.

The force on $q = q \vec{E}$ where $\vec{E}$ is the field due to the induced surface charge. But this is mimicked by the image charge for $z > 0$

$$\vec{F} = -\frac{1}{4\pi \varepsilon_0} \frac{q^2}{(2d)^2} \hat{z}$$

Work necessary to assemble charge distribution (not including the work in make a point charge)

$W = \text{Work necessary to move } q \text{ from } \infty \text{ to } z = d \text{ against the force of attraction to plane}$

$$W = -\int_{\infty}^{d} \vec{F} \cdot d\vec{l} = \frac{1}{4\pi \varepsilon_0} \int_{\infty}^{d} \frac{q^2}{z^2} \, dz$$

$$= -\frac{1}{4\pi \varepsilon_0} \left( \frac{-q^2}{d} \right)_{\infty}^{d} = -\frac{1}{4\pi \varepsilon_0} \frac{q^2}{4d}$$

Recall, for two point charges, the work required to assemble the distribution is

$$W = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{|r_1 - r_2|}$$

Thus, the work we found is $\frac{1}{2}$ that of assembling $q + \text{image}$. Reason: image comes for free on conductor
Example 2: Point charge brought near the surface of a grounded conducting sphere

By miraculous of miracles, it turns out that one can mimic the effects of the induced surface charge by a single image charge inside the sphere (see Problem Set)

On the surface of the sphere

\[ V(r) = \frac{1}{4\pi \epsilon_0} \left( \frac{Q}{r} + \frac{q_{\text{im}}}{2 \imath} \right) = 0 \]

Can do this by choosing

\[ q_{\text{im}} = -\frac{R}{s} Q \]
\[ Q = \frac{R^2}{s} \]

Force of attraction:

\[ F = \frac{1}{4\pi \epsilon_0} \frac{Q q_{\text{im}}}{(s-a)^2} = \frac{1}{4\pi \epsilon_0} \frac{Q^2 R s}{(s^2-R^2)^2} \]