Physics 405: Lecture 14

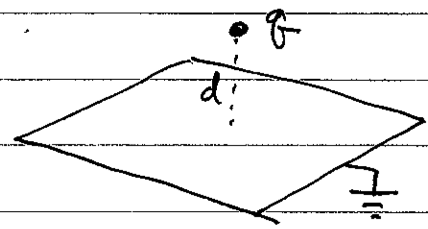
The Method of Images

The uniqueness ~~theorem~~ of the solutions to Poisson's Eqn for given boundary condition is a powerful result that gives rise to creative approaches. It implies that if we can "guess" a solution by any approach that satifies the boundary conditions, it must be *the* solution.

A beautiful example is the "method of images", used to find the solution for charges brought close to the surface of conductors. It is not really a "method"; it's more of an "art", involving some good intuition and creativity.

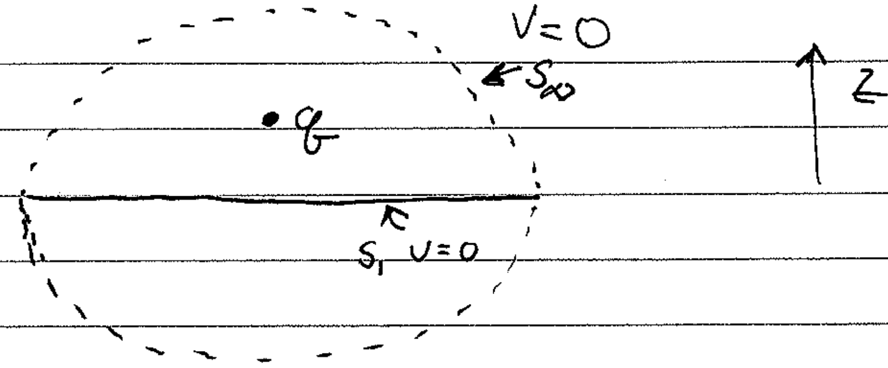
The "classic image problem"

A charge q is brought a distance d from an infinite ideal conducting plane. The conductor is grounded. Find the electric field everywhere



Physically, we know that oppositely charged surface charge density will be attracted to q, brought in from ground. But how is this charge distributed? And how to we find $\vec{E}$ from that charge configuration?

What are the boundary conditions?



(1) $V=0$ on plane $z=0$
(2) $V=0$ @ ∞
(3) Charge q @ $z=d$

We break up ~~the~~ space into two regions, $z>0$ and $z<0$. For $z<0$, V satifies Laplace's equation at $V=0$ on the surrounding boundary (plane + hemisphere $\to \infty$)

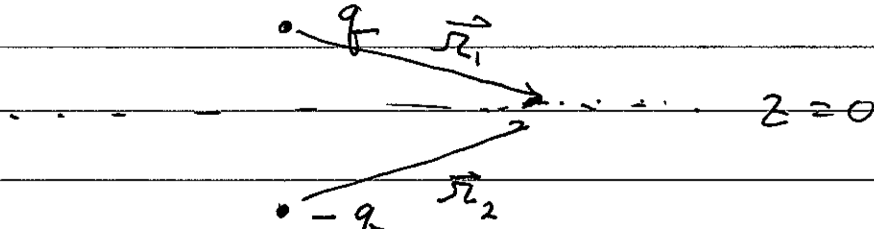
$$\Longrightarrow \boxed{V=0 \text{ everywhere for } z<0}$$

For $z>0$ V is a solution with charge @ $z=d$ and $V=0$ @ $z=0$ and $|\vec{r}| \to \infty$

Trick: In finding the solution for $z>0$, we can "mock up" the field generated by the surfaces charges on the plane by any charge distribution placed in the region $z<0$. If this distribution plus q produces a potential which satisfies all the boundary conditions the it is the solution for $z>0$.
Note: This does not give the solution for $z<0$.

"Dual" Problem: Find a charge distribution in $z<0$ which together with q @ $z=d$ make $V=0$ on conducting plane

Guess: An "image" charge of equal and opposite magnitude @ $z=-d$



$$V(x,y,z=0) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2}} - \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2}} = 0$$

at $V \Rightarrow 0$ @ $|\vec{r}| \Rightarrow \infty$

$\Rightarrow$ The image charge @ $z=-d$ mocks up the surface charges to produce the same field for $z>0$

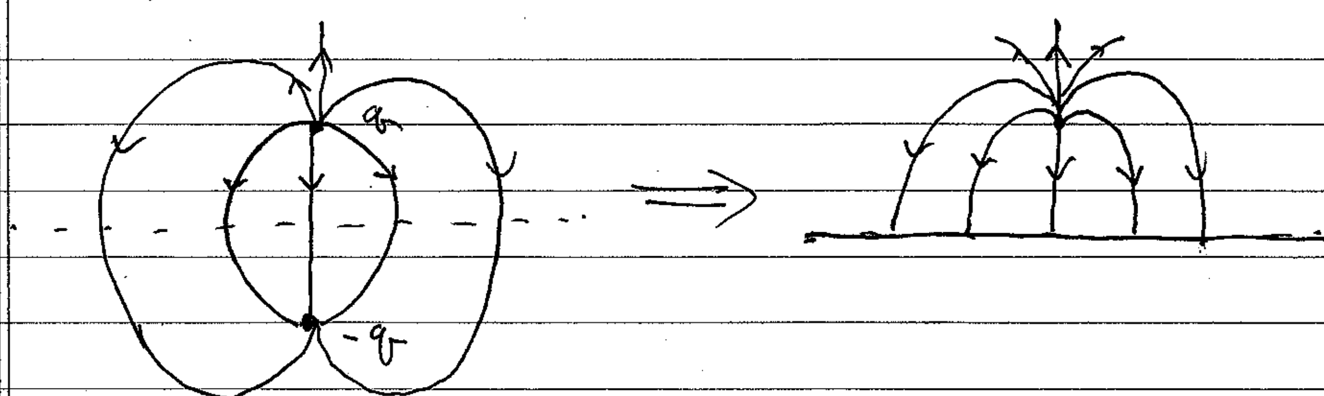


Image Problem

True probem

The Solution is thus

$$\boxed{V(x,y,z) = \begin{cases} 0 & z < 0 \\ \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right) & \text{for } z \geq 0 \end{cases}}$$

Given $V(\vec{r})$, we can find the true induced surface charge density based on the discontinuity relations of $\vec{E}_\perp$

Recall: $\Delta E_\perp = \sigma/\epsilon_0$ $\quad E_z^> = -\frac{\partial V}{\partial z}\Big|_{0_>}$, $\quad E_z^< = 0$

$$\Rightarrow \sigma = \epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0_>} = \frac{q}{4\pi}\left\{\frac{-(z-d)}{((x^2+y^2)+(z-d)^2)^{3/2}} - \frac{-(z+d)}{((x^2+y^2)+(z+d)^2)^{3/2}}\right\}\Big|_{z=0}$$

$$\Rightarrow \boxed{\sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}}$$ Axial symmetric

Total induced charge

$$q_{induced} = \int_{plane} \sigma\, da = \int_0^\infty 2\pi r\, dr\, \sigma(r)$$

no surprise

$$= -qd \int_0^\infty \frac{r\,dr}{(r^2+d^2)^{3/2}} = -qd\left[-\frac{1}{(r^2+d^2)^{1/2}}\right]_0^\infty = \boxed{-q}$$

Induced force

Because of the induced surface charge, the point charge q will be attracted to the plane.

The force on $q = q\vec{E}$ where $\vec{E}$ is the field due to the induced surface charge. But this is mimicked by the image charge for $z>0$.

$$\Rightarrow \quad \boxed{\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{z}}$$

Work necessary to assemble charge distribution (not including the work in make a point charge)

W = Work necessary to move q from ∞ to $z=d$ against the force of attraction to plane

$$W = -\int_{\infty}^{d} \vec{F}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}\,dz$$

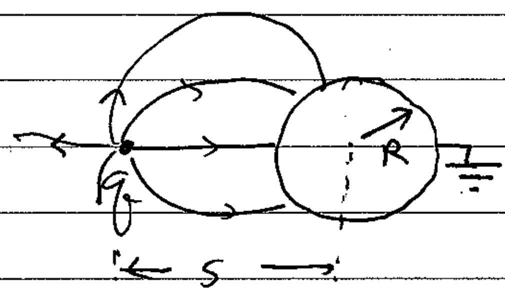
$$= \frac{1}{4\pi\epsilon_0}\left(\frac{-q^2}{4z}\right)_{\infty}^{d} = \boxed{-\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}}$$

Recall, for two points charges, the work required to assemble the distribution is

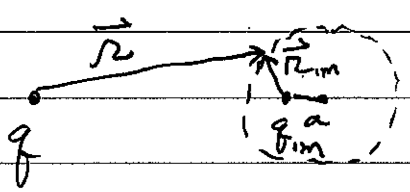
$$W = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

Thus, the work we found is $\frac{1}{2}$ that of assembling q + image. Reason: image comes for free on conductor

Example 2: Point charge brought near the surface of a grounded conducting sphere



By miricales of miricales, it turns out that one can mimick the effects of the induced surface charge by a single image charge inside the sphere (see Problem Set)



On the surface of the sphere

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\text{ɼ}} + \frac{q_{im}}{\text{ɼ}_{im}}\right) = 0$$

Can do this by choosing

$$\boxed{q_{im} = -\frac{R}{s}q}, \quad \boxed{a = \frac{R^2}{s}}$$ See Problem Set

Force of attraction: $F = \frac{1}{4\pi\epsilon_0}\frac{q\,q_{im}}{(s-a)^2} = \frac{-1}{4\pi\epsilon_0}\frac{q^2 R s}{(s^2-R^2)^2}$