Physics 405: Lecture 15

Solving Laplace's Eqn. by Separation of Variables

We seek the solution to Laplace's Eqn. \( \nabla^2 V = 0 \) subject to the boundary conditions. If the bounding surfaces are surfaces described by a fixed value of a coordinate (e.g. \( x, y, z \) or \( r, \theta, \phi \)) then we can employ the technique of "separation of coordinates" to solve the partial differential eqn.

Example: Griffies Ex 3.3

![Diagram showing three conducting strips with boundary conditions and differential equation setup.]

Three conducting strips:
- Two are grounded, separated by distance \( L \)
- Third has \( V(x, y) \)

Boundary conditions are independent of \( z \):

\[
\Rightarrow \frac{\partial^2 V}{\partial z^2} = 0 \Rightarrow V(x, y)
\]

Seek solution to \( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \)

subject to the boundary conditions:

1. \( V(x, 0) = 0 \)
2. \( V(x, L) = 0 \)
3. \( V(0, y) = f(y) \)
4. \( V(\infty, y) = 0 \)
Method of separation of variables

Given the Cartesian symmetry, we look for solutions of the form

\[ V(x, y) = X(x) Y(y) \]  \hspace{1cm} (Ansatz)

\[ \Rightarrow \nabla^2 V = \frac{\partial^2}{\partial x^2} (X(x) Y(y)) + \frac{\partial^2}{\partial y^2} (X(x) Y(y)) = 0 \]

\[ = Y(y) \frac{d^2 X}{dx^2} + X(x) \frac{d^2 Y}{dy^2} = 0 \]

\[ \Rightarrow \frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = 0 \]

\[ \frac{d^2 X}{dx^2} = F(x) \quad \frac{d^2 Y}{dy^2} = G(y) \]

\[ \Rightarrow F(x) + G(y) = 0 \hspace{1cm} \forall x \text{ and } \forall y \]

\[ \Rightarrow F(x) \text{ and } G(y) \text{ are constant} \]

\[ F(x) = K \quad G(y) = -K \Rightarrow \]

\[ \frac{d^2 X}{dx^2} = KX \]

Separation of the PDE into two ODE's of a solvable form.
We can easily to the solution to these differential equations by first defining $k^2 = k^2$ (note, if $k^2 < 0$, $k$ is imaginary).

\[
\frac{d^2 \chi}{dx^2} + k^2 \chi = 0 \quad \text{and} \quad \frac{d^2 \gamma}{dy^2} - k^2 \gamma = 0
\]

Two different sets of solutions:

1. $k$ real \( \Rightarrow \) \( k^2 > 0 \)
   \[ \chi(x) = A \sin kx + B \cos kx, \quad \gamma(y) = Ce^{ky} + De^{-ky} \]

2. $k$ imaginary \( \Rightarrow \) \( k^2 < 0 \)
   \[ \chi(x) = Ae^{kx} + Be^{-kx}, \quad \gamma(y) = Cs \sin ky + D \cos ky \]

The special solutions
\( (I) \)
\[ V_k(x, y) = (A \sin kx + B \cos kx)(Ce^{ky} + De^{-ky}) \]
\( (II) \)
\[ V_k(x, y) = (Ae^{kx} + Be^{-kx})(Cs \sin ky + D \cos ky) \]

are "normal modes" of the problem. The unknowns $A, B, C, D, k$ must be determined by the boundary conditions.

$B. C. I.$

$V(\infty, y) \to 0 \Rightarrow$ Must have solution of form $(II)$ with $A_k = 0$. Also, can absorb $B_k$ into $C_k$ and $D_k$ (two constants).

\[ \Rightarrow V_k(x, y) = e^{-kx} \left( C_k \sin ky + D_k \cos ky \right) \]
\[ B.C. \#1 \quad V_k(x,0) = 0 = D_k e^{-kx} \]

\[ \Rightarrow D_k = 0 \]

\[ \therefore V_k(x,y) = C_k e^{-kx} \sin k y \]

\[ B.C. \#2 \quad V_k(x,L) = 0 = C_k e^{-kx} \sin(kL) \]

\[ \Rightarrow \sin(kL) = 0 \quad \Rightarrow \text{Infinite number of possible solutions} \quad kL = n\pi \quad n = 1, 2, 3, 4, \ldots \]

Let \[ k_n = \frac{n\pi}{L} \]

\[ \Rightarrow V_n(x,y) = C_n e^{-k_n x} \sin k_n y \]

Discrete set of possible solutions

\[ B.C. \#3 \quad V(0,y) = f(y) \]

\[ V_n(0,y) = C_n \sin k_n y = f(y) \quad ?? \quad \text{Impossible unless f(y) is sine function} \]

But... Laplace's equation is linear \( \Rightarrow \) we can use the principle of superposition.

\[ \Rightarrow \text{The most general solution to Laplace's equation which satisfies b.c.'s 1, 2, and 4 is} \]

\[ V(x,y) = \sum_{n=1}^{\infty} C_n e^{-k_n x} \sin k_n y \]
We must thus find the coefficients \( C_n \) which satisfy B.C 1 

\[
V(x, y) = \sum_{n=1}^{\infty} C_n \sin(n k y) = f(y)
\]

**Fourier series.**

**Fourier's Theorem:** Any function on the interval \( 0 \rightarrow L \) can be expressed as superposition of sinusoidal functions with periods \( \frac{2 \pi}{n} \), \( n = 1, 2, \ldots \) with appropriate weights, \( C_n \).

**How do we find the \( C_n \)?**

**Trick:** Consider \( \int_0^L dy \sin(n k y) \sin(k m y) = \int_0^L dy \sin\left(\frac{n \pi y}{L}\right) \sin\left(\frac{m \pi y}{L}\right) \)

If \( n = m \): \( \int_0^L dy \sin^2\left(\frac{n \pi y}{L}\right) = \frac{L}{2} \) (area under positive curve)

If \( n \neq m \): \( \int_0^L dy \sin\left(\frac{n \pi y}{L}\right) \sin\left(\frac{m \pi y}{L}\right) = 0 \) (oscillated equally above and below axis)

\[ \Rightarrow \int_0^L dy \sin\left(\frac{n \pi y}{L}\right) \sin\left(\frac{m \pi y}{L}\right) = \frac{L}{2} \delta_{nm} \]

**"Kronecker delta"** \( \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \)
Thus, to satisfy b.c. #3

\[ V(x, y) = \sum_{n=1}^{\infty} C_n \sin k_n y = f(y) \]

Integrate both sides of the eqn w.r.t. \( \sin k_n y \) for an arbitrary index \( m = 1, 2, 3 \ldots k_m = m \frac{\pi}{L} \)

\[ \sum_{n=1}^{\infty} C_n \int_0^L \sin k_n y) \sin (k_m y) = \int_0^L f(y) \sin (k_m y) \]

\[ \frac{L}{2} \delta_{nm} \]

Sum over \( n \), not zero only when \( n = m \)

\[ \Rightarrow \quad C_m = \frac{2}{L} \int_0^L f(y) \sin (k_m y) \]

For example, suppose \( f(y) = V_0 \) (constant potential) relative to ground

\[ \Rightarrow \quad C_m = \frac{2}{L} V_0 \int_0^L \sin (k_m y) = \frac{2}{k_m L} V_0 (-\cos k_m y) \bigg|_0^L \]

\[ \Rightarrow \quad C_m = \frac{2}{k_m L} V_0 \left( 1 - \cos (m \pi) \right) \]

\[ \Rightarrow \quad C_m = \begin{cases} 0 & m \text{ even} \\ \frac{4V_0}{\pi m} & m \text{ odd} \end{cases} \]
The final solution is thus of the form

\[ V(x,y) = \frac{4V_0}{\Pi} \sum_{m = 1,3,5}^{\infty} \frac{1}{m} e^{-m^2 \frac{\Pi y}{L}} \sin \left( \frac{m \Pi y}{L} \right). \]

This sum can actually be performed to give a closed-form solution (this is rare)

\[ V(x,y) = \frac{2V_0}{\Pi} \tan^{-1}\left( \frac{\sin \left( \frac{\Pi y}{L} \right)}{\sinh \left( \frac{\Pi x}{L} \right)} \right). \]
Orthogonal functions

The example of Fourier series is a special example of so-called "orthogonal functions" which form a "basis" for the space of all "normalizable functions" on the interval.

Vector space with "inner product"

E.g. $\mathbb{R}^3 \quad \vec{A} = A_1 \vec{e}_1 + A_2 \vec{e}_2 + A_3 \vec{e}_3$

$\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ is a basis and "orthonormal"

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \quad \text{"complete set"}

To find expansion coefficients

$A_i = \vec{e}_i \cdot \vec{A} \quad \text{"project onto } \vec{e}_i \text{"} $

Space of functions on interval

Define "inner-product" of functions

$f \ast g = \int_{0}^{L} f(x) g(x) dx$

$\Rightarrow \text{"Norm of function" } \| f \| = \sqrt{\int_0^L f \ast f}$

"Normalizable" if $\| f \| < \infty$
\[ \text{The set } \{ e_n(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) \}_{n=1,3,5} \]

are "square normalizable" on the interval 0 to L and "orthonormal."

\[ e_n \cdot e_m = \int_0^L e_n(x) e_m(x) \, dx \]

\[ = \frac{2}{L} \int_0^L \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) \, dx \]

\[ = \frac{1}{2} \delta_{nm} \]

\[ e_n \cdot e_m = \delta_{nm} \]

General normalizable function can be expressed in terms of the "basis function" \[ \{ e_n(x) \} \] is a "complete set."

\[ f(x) = \sum_{n=1}^{\infty} f_n e_n(x) \]

\[ f_n = e_n \cdot f = \sqrt{\frac{2}{L}} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \]