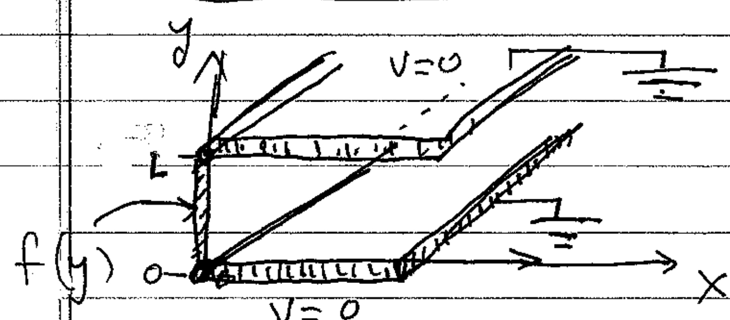
Physics 405: Lecture 15

Solving Laplace's Eqn by Separation of Variables

We seek the solution to Laplace's Eqn. $\nabla^2 V = 0$ subject to the boundary conditions. If the bounding surfaces are surfaces described by a fixed value of a coordinate (e.g. x, y, z or r, θ, ϕ) then we can employ the technique of "separation of coordinates" to solve the partial differential eqn.

Example: Griffiths Ex 3.3



Three conducting strips:
- Two are grounded, separated by distance L
- Third has $V(x,y)$

Boundary Conditions are Independent of z

$$\Rightarrow \frac{\partial V}{\partial z} = 0 \quad \Rightarrow \quad V(x,y)$$

Seek solution to $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

subject to the boundary conditions:

(1) $V(x,0) = 0$

(2) $V(x, L) = 0$

(3) $V(0,y) = f(y)$

(4) $V(\infty, y) = 0$

Method of separation of variables

Given the Cartesian symmetry, we look for solutions of the form

$$V(x,y) = X(x)\, Y(y) \qquad \text{(Ansatz)}$$

$$\Rightarrow \nabla^2 V = \frac{\partial^2}{\partial x^2}\left(X(x)\,Y(y)\right) + \frac{\partial^2}{\partial y^2}\left(X(x)\,Y(y)\right) = 0$$

$$= Y(y)\,\frac{d^2 X}{dx^2} + X(x)\,\frac{d^2 Y}{dy^2} = 0$$

$$\Rightarrow \underbrace{\frac{1}{X(x)}\,\frac{d^2 X}{dx^2}}_{F(x)} + \underbrace{\frac{1}{Y(y)}\,\frac{d^2 Y}{dy^2}}_{G(y)} = 0$$

$$\Rightarrow F(x) + G(y) = 0 \quad \forall x \text{ and } \forall y$$

$$\Longrightarrow \quad F(x) \text{ and } G(y) \text{ are } \underline{\text{constant}}$$

$$F(x) = K, \; G(y) = -K \;\Rightarrow\; \boxed{\begin{aligned} \frac{d^2 X}{dx^2} &= K X \\ \frac{d^2 Y}{dy^2} &= -K Y \end{aligned}}$$

Separation of the PDE into two ODE's of a solvable form.

We can easily to the solution to these diff'eqns by first defining $K \equiv k^2$ (note, if $K<0$, k imag)

$$\boxed{\frac{d^2 X}{dx^2} + k^2 X = 0} \; ; \quad \boxed{\frac{d^2 Y}{dy^2} - k^2 Y = 0}$$

Two different sets of solutn

(1) k real $\Rightarrow$ $k^2 > 0$

$$X(x) = A\sin kx + B\cos kx, \quad Y(y) = Ce^{ky} + De^{-ky}$$

(2) k imag $\Rightarrow$ $k^2 < 0$

$$X(x) = Ae^{kx} + Be^{-kx}, \quad Y(y) = C\sin ky + D\cos ky$$

The special solution $\overset{(I)}{V_k}(x,y) = (A_k \sin kx + B_k \cos kx)(C_k e^{ky} + D_k e^{-ky})$

or $\overset{(II)}{V_k}(x,y) = (A_k e^{kx} + B_k e^{-kx})(C_k \sin ky + D_k \cos ky)$

are "normal modes" of the problem. The unknowns, A, B, C, D, k must be determined by the boundary conditions.

B.C.'s

$V(\infty, y) \to 0 \Rightarrow$ Must have solution of form (II) with $A_k = 0$. Also, can absorb B_k into C_k and D_k (two constants)

$$\Rightarrow \boxed{V_k(x,y) = e^{-kx}(C_k \sin ky + D_k \cos ky)}$$

B.C. #1 $V_k(x,0) = 0 = D_k e^{-kx}$

$$\Rightarrow D_k = 0$$

$$\therefore \quad V_k(x,y) = C_k e^{-kx} \sin ky$$

B.C. #2 $V_k(x,L) = 0 = C_k e^{-kx} \sin(kL)$

$\Rightarrow \sin(kL) = 0 \Rightarrow$ Infinite number of possible solutions $kL = n\pi \quad n = 1, 2, 3, 4, \ldots \infty$

Let $\boxed{k_n = \frac{n\pi}{L}} \Rightarrow \boxed{V_n(x,y) = C_n e^{-k_n x} \sin k_n y}$

Discrete set of possible solutions

B.C. #3 $V(0,y) = f(y)$

$V_n(0,y) = C_n \sin k_n y = f(y)$?? Impossible unless $f(y)$ is sine function

But... Laplace's equation is linear $\Rightarrow$ we can use the principle of superposition.

$\Rightarrow$ The most general solution to Laplace's eqn which satisfies b.c.'s 1, 2, and 4 is

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n e^{-k_n x} \sin k_n y}$$

We must thus find the coefficients C_n which satisfy B.C #3

$$\boxed{V(0, y) = \sum_{n=1}^{\infty} C_n \sin(k_n y) = f(y)}$$

Fourier series!

Fourier's theorem: Any function on the interval $0 \to L$ can be expressed as superposition of sinusoidal functions with periods $\frac{2L}{n}$, $n = 1, 2 \cdots$ with appropriate weights, C_n.

How do we find the C_n?

Trick: Consider $\int_0^L dy \sin(k_n y) \sin(k_m y) = \int_0^L dy \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi y}{L}\right)$

If $n = m$ $\quad \int_0^L dy \sin^2\left(\frac{n\pi y}{L}\right) = \frac{L}{2}$ (area under positive curve)

If $n \neq m$ $\quad \int_0^L dy \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi y}{L}\right) = 0$ (oscillates equally above and below axis.)

$$\Rightarrow \boxed{\int_0^L dy \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi y}{L}\right) = \frac{L}{2}\delta_{nm}}$$

"Kronecker delta" $\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

Thus, to satisfy b.c. #3

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin k_n y = f(y)$$

Integrate both sides of the eqn w.r.t $\sin k_m y$ for an arbitrary index $m = 1, 2, 3 \ldots$ $k_m = m\pi/L$

$$\Rightarrow \sum_{n=1}^{\infty} C_n \underbrace{\int_0^L dy \sin(k_n y) \sin(k_m y)}_{\frac{L}{2}\delta_{nm}} = \int_0^L f(y) \sin(k_m y)$$

Sum over n, not zero only when $n = m$

$$\Rightarrow \boxed{C_m = \frac{2}{L}\int_0^L dy\, f(y) \sin(k_m y)}$$

For example, suppose $f(y) = V_0$ (constant potential relative to ground)

$$\Rightarrow C_m = \frac{2V_0}{L}\int_0^L dy \sin(k_m y) = \frac{2}{k_m L} V_0 \left(-\cos k_m y\right)_0^L$$

$$\Rightarrow C_m = \frac{2}{\pi L} V_0 \left(1 - \cos(m\pi)\right)$$

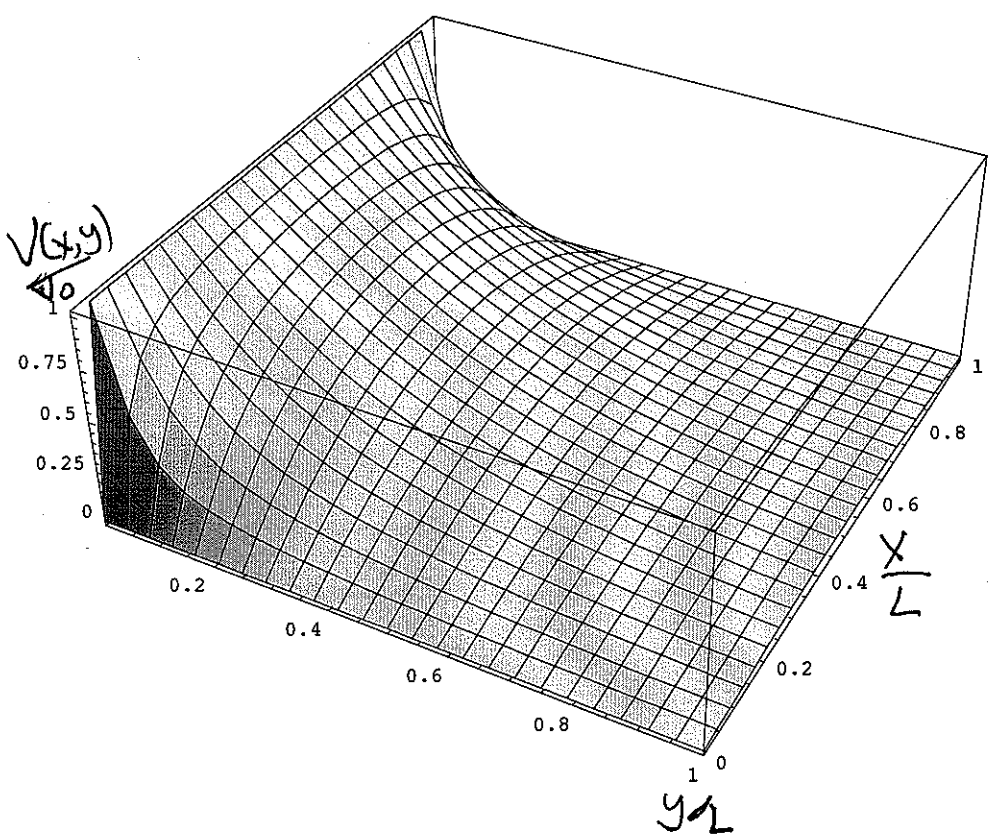
$$\Rightarrow C_m = \begin{cases} 0 & m \text{ even} \\ \frac{4V_0}{\pi m} & m \text{ odd} \end{cases}$$

The final solution is thus of the form

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{m=1,3,5}^{\infty} \frac{1}{m} e^{-\frac{m\pi x}{L}} \sin\left(\frac{m\pi y}{L}\right)}$$

This sum can actual be performed to give a closed-form solution (this is rare)

$$\boxed{V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin\left(\frac{\pi y}{L}\right)}{\sinh\left(\frac{\pi x}{L}\right)}\right)}$$



Orthogonal functions

The example of Fourier series is a special example of so-called "orthogonal functions" which form a "bases" for the space of all "normalizable functions" on the interval

Vector space with "inner product"

E.g. $\mathbb{R}^3$ $\quad \vec{A} = A_1 \vec{e}_1 + A_2 \vec{e}_2 + A_3 \vec{e}_3$

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis and "orthonormal"

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ "complete set"

To find expansion coefficients

$A_i = \vec{e}_i \cdot \vec{A}$ "Project onto $\vec{e}_i$"

Space of functions on interval

Define "inner-product" of functions

$$f \cdot g \equiv \int_0^L dx\, f(x)\, g(x)$$

$\Rightarrow$ "Norm of function" $\|f\| = \sqrt{f \cdot}$

"Normalizable" if $\|f\| < \infty$

$\Rightarrow$ The set $\left\{ e_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right.$ $n = 1, 2, 3 \ldots$

are "square normalizable" on the interval $0 \to L$
and "orthornomal"

$$e_n \bullet e_m = \int_0^L e_n(x)\, e_m(x)\, dx$$

$$= \frac{2}{L} \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{\frac{L}{2}\delta_{nm}}$$

$$\Rightarrow \boxed{e_n \bullet e_m = \delta_{nm}}$$

General normalizable function can be expressed in terms of the "basis function"
$\{e_n(x)\}$ is a "complete set"

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} f_n\, e_n(x)$$

$$\Rightarrow \boxed{f_n = e_n \bullet f = \sqrt{\frac{2}{L}} \int_0^L dx\, f(x) \sin\left(\frac{n\pi x}{L}\right)}$$