Physics 405 Lecture 16

Separation of Variables in Spherical Coords

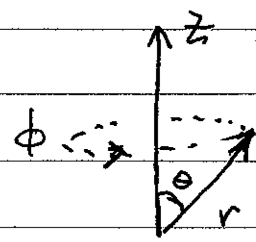
As discussed in Lect. 15, the method of separation of variables is useful when the boundaries lie on surfaces of constant coordinates. Cartesian coords are applicable when the boundaries are along x, y, z. In many more "natural" situations, boundaries lie on surfaces of constant r, θ, ϕ

Laplace equation in spherical coords.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$= 0$$

We will restrict our attention to problems with azimuthal symmetry, no variation with ϕ



$$\frac{\partial V}{\partial \phi} = 0$$

2D Laplace

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0}$$

As in Lect. 15 we assume a form (ansatz) in which the potential "separates" into a product of functions, each depending only on one coordinate

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

Plugging into laplace's eqn

$$\Rightarrow \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{F(r)} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{G(\theta)} = 0$$

$$\Rightarrow F(r) = -G(\theta) = K \quad \text{(separation constant)}$$

Radial eqn:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = K R(r)$$

Trial solution: $R(r) = A r^\ell$

$$\Rightarrow \frac{d}{dr}\left(\ell r^{\ell+1}\right) = K r^\ell \Rightarrow \ell(\ell+1) r^\ell = K r^\ell$$

$$\Longrightarrow K = \ell(\ell+1) \quad \text{Yes}$$

But second order ODE $\Rightarrow$ Two independent solutions

Other: $R(r) = \dfrac{B}{r^{\ell+1}}$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = B\frac{d}{dr}\left(-(\ell+1)\, r^{-\ell}\right) = \ell(\ell+1) R$$

$\Rightarrow$ General radial solution

$$R_\ell(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \qquad \ell = 0, 1, 2 \ldots$$

Angular Solution

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{\partial \Theta}{\partial\theta}\right) = -\ell(\ell+1)\,\Theta(\theta)$$

When dealing with θ-direction it is useful to define $\quad \mu \equiv \cos\theta \quad \Rightarrow d\mu = -\sin\theta\, d\theta$

$$\frac{d}{d\theta} = \frac{d\mu}{d\theta}\frac{d}{d\mu} = -\sin\theta \frac{d}{d\mu} \Rightarrow \frac{d}{d\mu} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$$

Let $P(\cos\theta) = \Theta(\theta)$

$$\Rightarrow \frac{d}{d\mu}\left(+\sin^2\theta \frac{dP}{d\mu}\right) = -\ell(\ell+1)P(\mu)$$

$$\boxed{\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP}{d\mu}\right] = -\ell(\ell+1)P(\mu)}$$

Legendre's Eqn.

One Solution is a Polynomial of order ℓ

Legendre Polynomial: $\displaystyle P_\ell(\mu) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{d\mu^\ell}(\mu^2-1)^\ell$

(Rodrigues's formula)

$$P_0(\mu) = 1 \quad \Rightarrow \quad \Theta_0(\theta) = 1$$

$$P_1(\mu) = \mu \quad \Rightarrow \quad \Theta_1(\theta) = \cos\theta$$

$$P_2(\mu) = \frac{3}{2}\mu^2 - \frac{1}{2} \quad \Rightarrow \quad \Theta_2(\theta) = \frac{3\cos^2\theta - 1}{2} \quad \ldots$$

$$\vdots \qquad\qquad\qquad \vdots$$

Note: The factor $\frac{1}{2^\ell \ell!}$ is a convention so that

$$P_\ell(1) = 1 \quad \forall \ell$$

Second solution:

Can ~~sol~~ show $\Theta_\ell(\theta) = Q_\ell(\mu)$

where $Q_0(\mu) = \frac{1}{2}\ln\left[\frac{1-\mu}{1+\mu}\right]$, $Q_1(\mu) = \ldots$

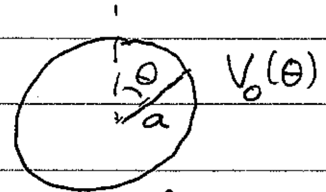
Note Q_0 blows up at $\mu = 1$ and -1 $\Rightarrow \theta = 0, \pi$

$\Rightarrow$ Not a physically allowed solution unless region excludes $\theta = 0$ and $\theta = \pi$ = Rare

$\Rightarrow$ General solution

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)}$$

Example: On the surface of a sphere of radius "a" the potential is specified a $V_0(\theta)$



Find $V(\vec{r})$ everywhere in space.

Boundary Conditions

- V does not blow up anywhere
- $V(r=a, \theta) = V_0(\theta)$
- $V(\vec{r})$ continuous

General solution $V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

Consider two regions (I) $r<a$, (II) $r>a$

(I) Inside sphere $r<a$

Must have $B_\ell = 0 \ \forall \ell$, otherwise solution blows up.

At boundary, $V(a,\theta) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta) = V_0(\theta)$

To find the coefficients A_ℓ we use the fact that the Legendre polynomial are a "complete basis" for the space of functions on the interval $-1 \le \cos\theta \le +1$

i.e. $\pi \le \theta < 0$

Fact: (we won't prove here)

$$\boxed{\int_0^{\pi} P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)\,\sin\theta\, d\theta = -\int_0^{\pi} P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)\, d(\cos\theta) = \int_{-1}^{1} P_\ell(\mu)\, P_{\ell'}(\mu)\, d\mu = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}}$$

Note: The Legendre Polynomials are not normalized in the standard convention.

Given this fact, we can find the expansion coefficients, A_ℓ.

$$\sum_{\ell'=0}^{\infty} A_{\ell'}\, a^{\ell'} P_{\ell'}(\cos\theta) = V_0(\theta)$$

dummy variable

Integrate both sides $\int_0^{\pi} \sin\theta\, d\theta\, P_\ell(\cos\theta)$

$$\Rightarrow \sum_{\ell'=0}^{\infty} A_{\ell'}\, a^{\ell'} \left(\frac{2}{2\ell+1}\right) \delta_{\ell\ell'} = \int_0^{\pi} \sin\theta\, d\theta\; P_\ell(\cos\theta)\, V_0(\theta)$$

$$\Rightarrow \boxed{A_\ell = \frac{2\ell+1}{2a^\ell} \int_0^{\pi} V_0(\theta)\, P_\ell(\cos\theta)\, \sin\theta\, d\theta}$$

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II <u>Outside sphere:</u> $r > a$

Must have $V \to 0$ as $r \to \infty$

$$\Rightarrow V_{II}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Boundary condition $V(\vec{r})$ continuous

$$\Rightarrow V_I(a,\theta) = V_{II}(a,\theta)$$

$$\Rightarrow \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta)$$

By orthogonality of the Legendre Polynomials

$$\Rightarrow A_l a^l = B_l a^{-(l+1)}$$

$$\Rightarrow B_l = a^{2l+1} A_l$$

$\therefore$ General solution given $V(a,\theta) \equiv V_0(\theta)$

$$\boxed{V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & r \le a \\ \sum_{l=0}^{\infty} A_l \frac{a^{2l+1}}{r^{l+1}} P_l(\cos\theta) & r \ge a \end{cases}}$$

where $A_l = \frac{2l+1}{2a^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$

Though the solution is general, often we can "eyeball" the coefficients A_ℓ without performing an integral.

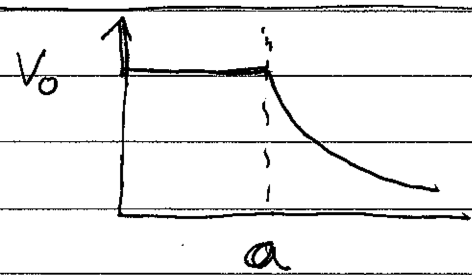
<u>Example:</u> Suppose $V_0(\theta) = V_0$ <u>constant</u>

Since $P_0(\cos\theta) = 1 \Rightarrow A_\ell = 0 \quad \forall \ell > 0$

$$\Rightarrow V_0(\theta) = V_0 \, P_0(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell a^\ell r^\ell$$

$$\Rightarrow A_0 = V_0 , \quad A_\ell = 0 \quad \ell > 0$$

$$\therefore \boxed{V(r) = \begin{cases} V_0 & r \leq a \\ \dfrac{V_0 a}{r} & r \geq a \end{cases}}$$



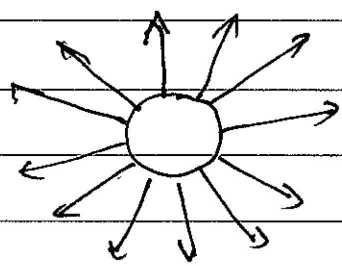
This is exactly what we would expect since a spherically symmetric charge distribution only a spherical shell makes that shell an equipotential surface. Inside the shell V is constant. Outside the shell it is as if we had a point charge at origin $V \sim \frac{1}{r}$

Electric Field:

$$\vec{E} = -\vec{\nabla} V = -\hat{r}\frac{\partial V}{\partial r} - \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \theta}$$

Here V depends only on r

$$\Rightarrow \vec{E} = \begin{cases} 0 & r \le a \\ +\dfrac{V_0 a}{r^2}\hat{r} & r \ge a \end{cases}$$



Discontinuity in $\vec{E}$ at radial shell $\Rightarrow$ surface charge at the radius a

$$\Delta E_r \Big|_a = \frac{\sigma}{\epsilon_0}(r=a)$$

$$\Rightarrow \sigma = \epsilon_0 \frac{V}{a} = \frac{Q_{tot}}{4\pi a^2}$$

$$\Rightarrow Q_{tot} = 4\pi\epsilon_0 V_0 a$$

Check: $\vec{E}(\vec{r}) = \dfrac{Q_{tot}}{4\pi\epsilon_0 r^2}\hat{r} = \dfrac{V_0 a}{r^2}\hat{r}$ ✓

Example: $V_0(\theta) = V_0 \sin^2\frac{\theta}{2}$

Note: Using trig identities

$$V_0(\theta) = V_0\left(\frac{1-\cos\theta}{2}\right) = \frac{V_0}{2}P_0(\cos\theta) - \frac{V_0}{2}P_1(\cos\theta)$$

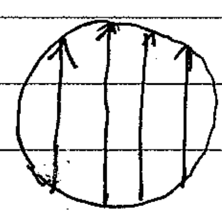
$$= \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta)$$

$$\Rightarrow A_0 = \frac{V_0}{2} \qquad aA_1 = -\frac{V_0}{2} \Rightarrow A_1 = -\frac{V_0}{2a}$$

$$A_\ell = 0 \text{ for } \forall \ell > 1$$

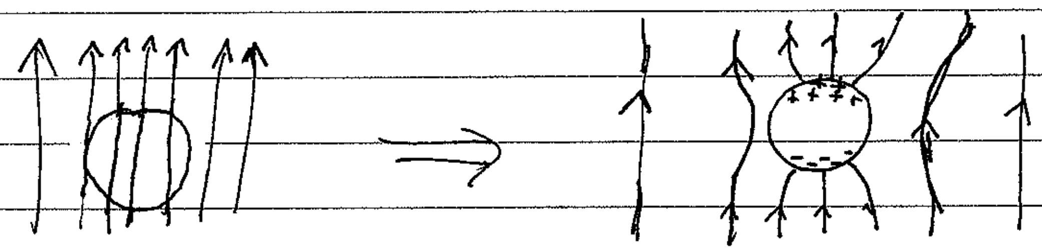
$$\therefore V(r,\theta) = \begin{cases} \frac{V_0}{2}\left(1 - \frac{r}{a}\cos\theta\right) = \frac{V_0}{2}(1-z) & r \leq a \\ \frac{V_0}{2}\left(\frac{a}{r} - \frac{a^3}{r^2}\cos\theta\right) & r \geq a \end{cases}$$

Note: Inside, E field is constant



Outside E field dominated by $\frac{a^3}{r^3}$ for $r \sim a$; $\frac{a}{r^2}$ for $r \to \infty$

Example: A conducting sphere radius a is placed in a uniform electric field $\vec{E} = E_0 \hat{z}$. Find the total field, including that of the induced surface charges



~~the~~ Boundary conditions

(i) $E = 0$ for $r \leq a$

(ii) $E \Rightarrow E_0 \hat{z}$ for $r \to \infty$ (since field of induced charges $\to 0$)

We can translate this into boundary conditions on the potential in the following way. We know that the surface of the conductor is an equipotential surface $\Rightarrow V = V_0$ on sphere. We might as well take that as ground

$\Rightarrow$ (i) $V = 0$ $\quad r = a$

(ii) $V = -E_0 z + C$ $\quad r \to \infty$

$= -E_0 r \cos\theta + C$

Azimuthal symmetry $\Rightarrow$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Inside sphere

$V = 0$ since $V=0$ on surface

Outside sphere

$$V(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

- On surface $r=a$

$$V(a,\theta) = \sum_\ell \left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} \right) P_\ell(\cos\theta) = 0$$

$$\Rightarrow B_\ell = -A_\ell a^{2\ell+1}$$

$$\Rightarrow V(r,\theta) = \sum_\ell A_\ell \left(r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

- At ∞

$$V(r\to\infty,\theta) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r\cos\theta$$

$$= -E_0 r P_1(\cos\theta)$$

$\Rightarrow A_1 = -E_0$; all other coefficents $= 0$

$$\Rightarrow V(r,\theta) = -E_0 \left(r - \frac{a^3}{r^2} \cos\theta \right)$$

$$= \underbrace{-E_0 z}_{\text{external field}} + \underbrace{\frac{E_0 a^3}{r^2} \cos\theta}_{\text{Induced field of surface charges}}$$

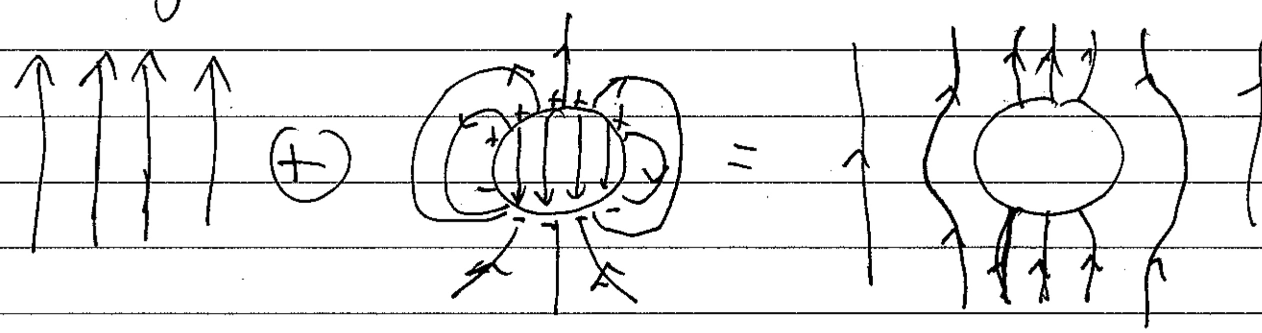
The induced surface charge

$$\sigma(\theta) = \epsilon_0 E_r(r=a,\theta) = \epsilon_0 \frac{\partial U}{\partial r}\Big|_{r=a}$$

$$= -\epsilon_0 E_0 \left(1 + 2\frac{a^3}{r^3}\right)\cos\theta \Big|_{r=a}$$

$$= 3\epsilon_0 E_0 \cos\theta$$

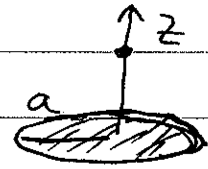
Note: The $\cos\theta$ distribution of surface charge induces a field which is uniform and opposite to the applied field inside the sphere, thereby cancelling the field in the conductor



Example: Azimuthal symmetry and boundary condition at $\theta = 0$.

Our solution to Laplace's Eqn in spherical coordinates in useful for problems of localized charge distributions that have axial (azimuthal) symmetry (not necessarily) spherical.

E.g. Disk of radius a, surface charge σ



Using the axial symmetry, we found V on z-axis (~~the~~ ground @ ∞)

$z>0$
$$V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+a^2} - z\right)$$

Using spherical coordinates, z-axis ($z>0$) is $\theta > 0$ $z = r$

$$\Rightarrow V(r, \theta=0) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+a^2} - r\right)$$

We now outside the charge

$$V(r,\theta) = \sum_{\ell=0}^{\infty}{}' \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$\Rightarrow V(r, \theta=0) = \sum_{\ell=0}^{\infty}{}' \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) \underbrace{P_\ell(1)}_{=1}$$

The solution on the z-axis thus acts as a boundary condition @ $\theta = 0$

Consider region $r > a \Rightarrow \frac{a}{r} < 1$

$$V(r, \theta=0) = \frac{\sigma r}{2\epsilon_0}\left[\left(1 + \frac{a^2}{r^2}\right)^{1/2} - 1\right]$$

We can Taylor expand

$$\left(1 + \frac{a^2}{r^2}\right)^{1/2} = 1 + \frac{a^2}{2r^2} - \frac{1}{8}\frac{a^4}{r^4} + \cdots$$

$\Rightarrow$ For $r > a$ $$V(r, \theta=0) = \frac{\sigma}{2\epsilon_0}\left(\frac{a^2}{2r} - \frac{a^4}{8r^3} + \cdots\right)$$

$$= \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)$$

$$\Rightarrow A_\ell = 0 \quad \forall \ell, \qquad B_0 = \frac{\sigma a^2}{4\epsilon_0} = \frac{\pi a^2 \sigma}{4\pi\epsilon_0} = \frac{Q}{4\pi\epsilon_0} \checkmark$$

$$B_2 = \frac{\sigma a^4}{16\epsilon_0} = \frac{-\frac{Qa^2}{4}}{4\pi\epsilon_0} \leftarrow \text{Quadrupole}$$

$B_\ell = 0$ ℓ odd

$$r > a \Rightarrow \boxed{V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} - \frac{\frac{1}{4}Qa^2}{r^3} P_2(\cos\theta) + \cdots\right]}$$

The dominate term is $\frac{Q}{r}$ ← point charge

the next correction is "quadrupole" — next lecture