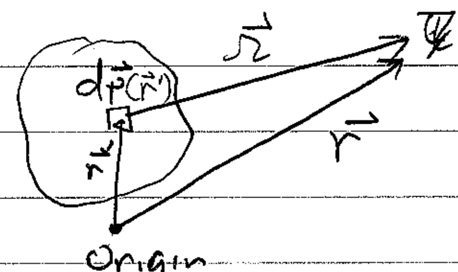
Physics 405: Lecture 20

Electrostatics in Dielectrics

Field of a polarized object

Given a material with polarization $\vec{P}(\vec{r})$, what is the electric field it generates (not to be confused with an external field which may have induced $\vec{P}$).



In every "chunk" $d\vec{r}'$ there is a local dipole

$$d\vec{p}(\vec{r}') = \vec{P}(\vec{r}')\, d^3r'$$

The potential of a dipole $dV(\vec{r}) = \dfrac{\hat{\mathscr{r}} \cdot d\vec{p}(\vec{r}')}{4\pi\epsilon_0 \mathscr{r}^2}$

$$\Rightarrow \boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \vec{P}(\vec{r}')}$$

Now we use a trick:

Note $\vec{\nabla}' \frac{1}{\mathscr{r}} = \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|} = -\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|}$ (chain rule)

$$= \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\therefore V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{Vol} d^3r' \left(\vec{\nabla}' \frac{1}{\mathscr{r}}\right) \cdot \vec{P}(\vec{r}')$$

Using integration by parts, we find

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{Vol} d^3r' \left(\frac{1}{\mathscr{r}}\right)\left(-\vec{\nabla}' \cdot \vec{P}(\vec{r}')\right) + \oint_{\substack{\text{Surface}\\ \text{bounding Vol}}} \frac{\vec{P}(\vec{r}') \cdot d\vec{a}'}{\mathscr{r}} \right]$$

We thus see that we can write $V(\vec{r})$ as

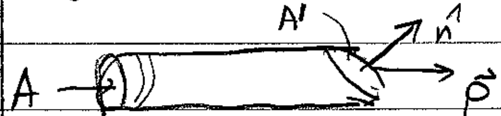
$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{Vol} \frac{\rho_b(\vec{r}')\, d^3r'}{|\vec{r} - \vec{r}'|} + \oint_{surface} \frac{\sigma_b(\vec{r}')\, da'}{|\vec{r} - \vec{r}'|}}$$

Where $\boxed{\rho_b(\vec{r}) = -\vec{\nabla} \cdot \vec{P}(\vec{r}) \equiv \text{Bound charge density}}$

$$\boxed{\sigma_b(\vec{r}) = \hat{n} \cdot \vec{P}(\vec{r}) = \text{Bound surface charge density}}$$

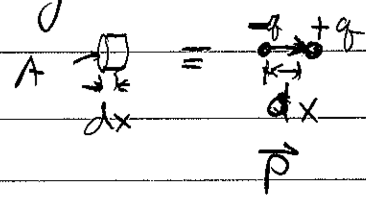
($\hat{n}$: normal to surface)

Physical Interpretation



- Consider uniformly polarized rod ($\vec{P} \parallel$ rod)

For every chunk



$|\vec{p}| = |\vec{P}| A\,dx = q\,dx$

$\Rightarrow P = \frac{q}{A}$ (surface charge)

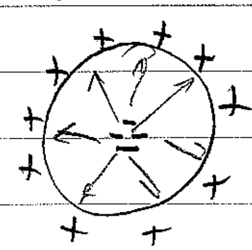
$$\hat{n} \cdot \vec{P} = P\cos\theta = \frac{q}{A/\cos\theta} = \frac{q}{A'} = \sigma_b \checkmark$$

- When the polarization is not uniform, we accumulate bound charge within the volume, as well as at the surface. Since the material is neutral, the total bound surface charge must be equal and opposite to the bound volume charge

$$\int_{Vol} \rho_b \, d^3r = -\oint_{Surface} \vec{P} \cdot \hat{n} \, da = \int_{Vol} \left(-\vec{\nabla} \cdot \vec{P}\right) d^3r$$

$$\Rightarrow \rho_b(\vec{r}) = -\vec{\nabla} \cdot \vec{P}(\vec{r})$$

Example: radially directed $\vec{P} = P(r)\,\hat{r}$



Example: Uniformly polarized sphere $\vec{P} = P_0 \hat{z}$

To find the field everywhere, we find the bound charge distribution associated with the polarized object

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$$

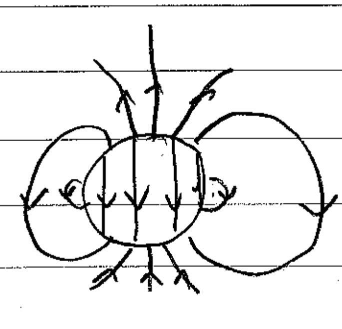
$$\sigma_b = \hat{r} \cdot \vec{P} = P_0 \cos\theta \quad \text{on surface.}$$

But we solved this problem in Lect. 18.
For a surface charge density $\sigma(\theta) = \sigma_0 \cos\theta$, the dipole moment is $p = \frac{4\pi}{3} R^3 \sigma_0$

Here $\sigma_0 = P_0 \Rightarrow p = \frac{4\pi}{3} R^3 P_0$ = total dipole moment!

This surface charge produces a potential

$$V(r) = \begin{cases} \dfrac{P_0}{3\epsilon_0} z & r < R \\[2ex] \dfrac{p\cos\theta}{4\pi\epsilon_0 r^2} = \dfrac{P_0 R^3}{3\epsilon_0 r^2} & r > R \end{cases}$$



Electric Displacement

At times we are interested in the field produced by the free charges we apply in the presence of a dielectric medium; e.g. a charged plate in air. The electric field is due to the total charge, bound and free

$$\rho = \rho_b + \rho_f$$

According to Gauss's Law

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho = \rho_b + \rho_f = -\vec{\nabla} \cdot \underline{P} + \rho_f$$

$$\Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

We define the "Electric Displacement" field

$$\vec{D}(\vec{r}) = \epsilon_0 \vec{E}(\vec{r}) + \vec{P}(\vec{r})$$

Gauss's Law for the "macroscopic field" in dielectrics is

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f}$$

or in integral form

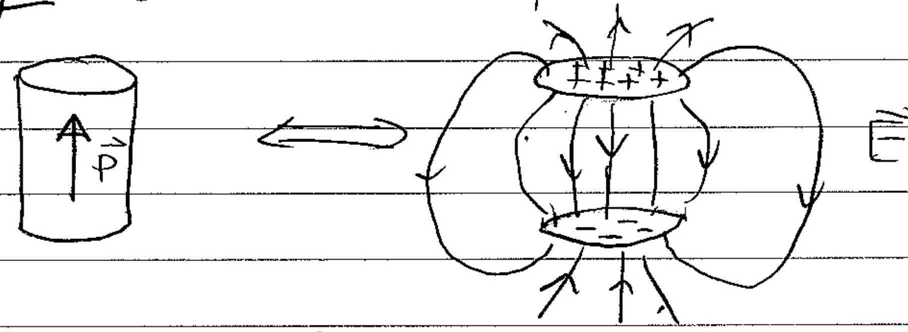
$$\boxed{\oint_S \vec{D} \cdot da = Q_{enc}^{free}}$$

Note: $\vec{\nabla}\times\vec{D} = \epsilon_0 \cancelto{0}{\vec{\nabla}\times\vec{E}} + \vec{\nabla}\times\vec{P}$

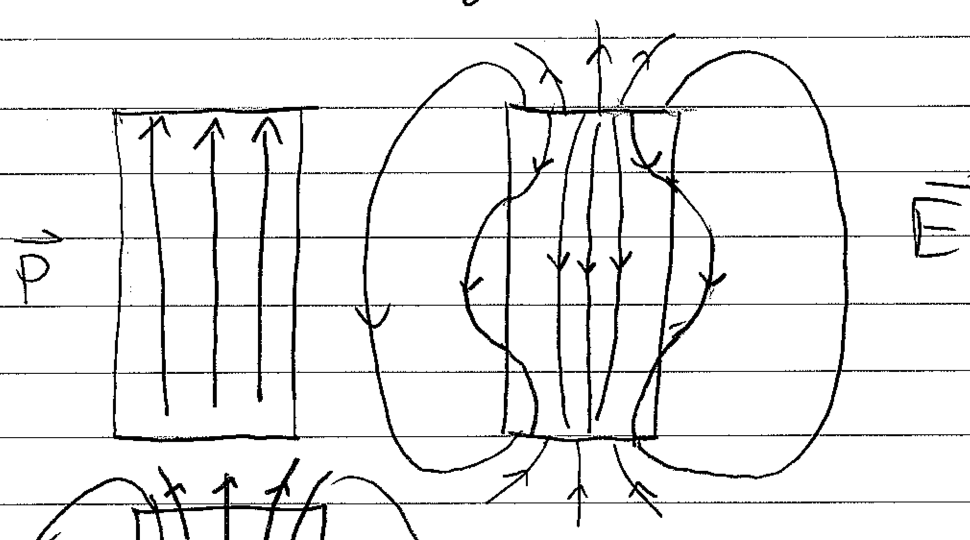
$\vec{\nabla}\times\vec{P}$ is not necessarilly zero

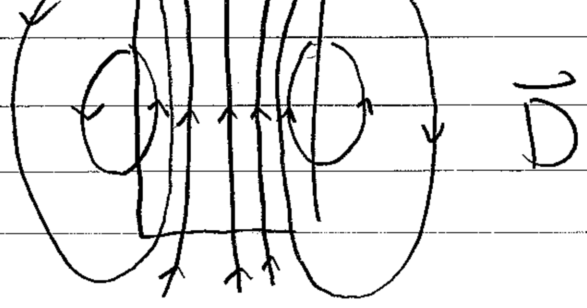
So, generally, $\vec{D}$ is not determinied solely by ρ_{free} as $\vec{E}$ is determend by ρ_{total}

Example: Bar electret (permanent electric dipole aligment)



More Preasely





Note: Here
$\vec{\nabla}\cdot\vec{D} = 0$

Linear Dielectrics

For most materials, and moderate applied fields, the magnitude on the polarization is proportional to the magnitude of $\vec{E}$. Such dielectric response is said to be "linear". For isotropic media

$$\vec{P} = \epsilon_0 \underset{\uparrow}{\chi} \vec{E}$$

dielectric susceptibility

$\Rightarrow$ For linear dielectric $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$$\Rightarrow \vec{D} = \epsilon_0 (1+\chi) \vec{E}$$

$$\equiv \epsilon \vec{E}$$

$\epsilon \equiv \epsilon_0 (1+\chi)$ = "dielectric permitivity"

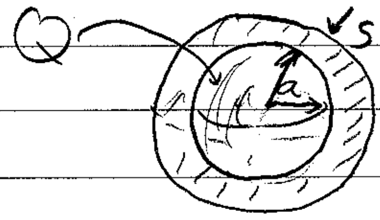
$k \equiv \dfrac{\epsilon}{\epsilon_0}$ = "dielectric constant"

Note: $\vec{\nabla} \times \vec{D} = \epsilon \vec{\nabla} \times \vec{E} = 0$ (in electrostatics for linear media) everywhere except at surface

$\Rightarrow \vec{D}$ is completely determined by the free charges everywhere except at the boundary

Example: A sphere carrying surface charge Q, with uniformly distribution, is surrounded by an insulating shell of thickness S. The dielectric permittivity of the shell is ϵ.

Find the potential at the center with ground @ infinity



The potential is defined relative to the force exerted, which is defined by the $\vec{E}$-field

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

To find the $\vec{E}$-field, use the fact that

$$\vec{D} = \epsilon \vec{E}$$

$\Rightarrow$ Inside sphere $r < a$ $\quad \vec{D} = \vec{E} = 0$

- Inside dielectric $a < r < a+s$ $\quad \vec{E}(\vec{r}) = \dfrac{\vec{D}(\vec{r})}{\epsilon}$

- Outside material $r > a+s$ $\quad \vec{E}(\vec{r}) = \dfrac{\vec{D}(\vec{r})}{\epsilon_0}$

To find $\vec{D}$, we can use Gauss's Law and symmetry

$$\oint \vec{D} \cdot d\vec{a} = Q_{enc}$$

And since $\vec{\nabla} \times \vec{D} = 0$ $\quad \vec{D}(\vec{r}) = D(r)\,\hat{r}$

$$\Rightarrow D(r) = \begin{cases} 0 & r < a \\ \frac{Q}{4\pi r^2} & r > a \end{cases}$$

$$\Rightarrow \vec{E}(\vec{r}) = E(r)\,\hat{r} \qquad E(r) = \begin{cases} 0 & r < a \\ \frac{Q}{4\pi\epsilon r^2} & a < r < a+s \\ \frac{Q}{4\pi\epsilon_0 r^2} & r > a+s \end{cases}$$

$$\Rightarrow V(0) = \int_0^\infty E(r)\,dr = \frac{Q}{4\pi}\left[\int_a^{a+s} \frac{dr}{\epsilon r^2} + \int_{a+s}^{\infty} \frac{dr}{\epsilon_0 r^2}\right]$$

$$\boxed{V(0) = \frac{Q}{4\pi}\left[\frac{1}{\epsilon a} - \frac{1}{\epsilon(a+s)} + \frac{1}{\epsilon_0(a+s)}\right]}$$

Aside:

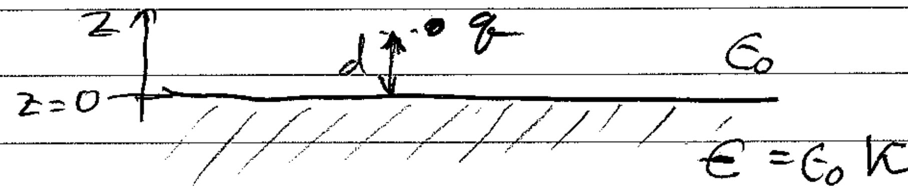
$a < r < a+s$ $\quad \vec{P}(\vec{r}) = \epsilon_0 \chi \vec{E}(\vec{r}) = \frac{\epsilon_0 \chi}{4\pi\epsilon} \frac{Q}{r^2}\,\hat{r}$

$$= \frac{\chi}{(1+\chi)} \frac{Q}{4\pi r^2}\,\hat{r}$$

$\vec{\nabla} \cdot \vec{P} = 0 \Rightarrow$ No bulk bound charge density ρ_b

Image Problem with Dielectrics

Consider a point charge q placed a distance d above the surface of a dielectric, with dielectric constant K. With what force will it be attracted to the surface?



Like the problem near a conductor, surface charges will be induced that attract q. Unlike the conductor, the field inside the dielectric is not zero. Nonetheless, we can mimick the effect of the surface charge through "image charge".

For $z>0$, consider

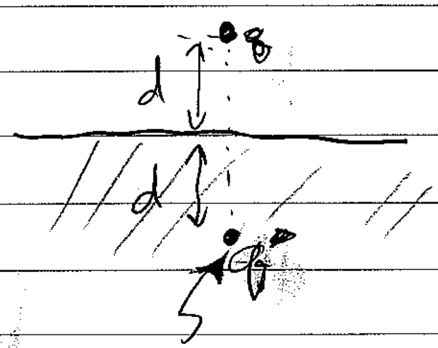


Image charge mimicking the effect of surface charge on $\vec{E}$ for $z>0$

$z<0$, consider

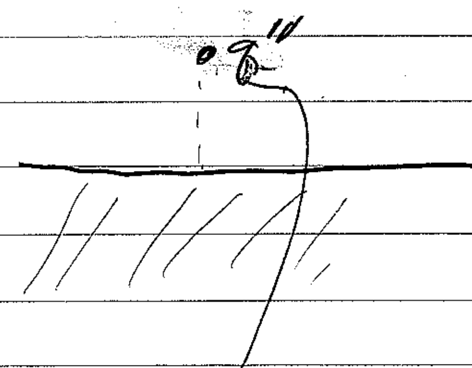
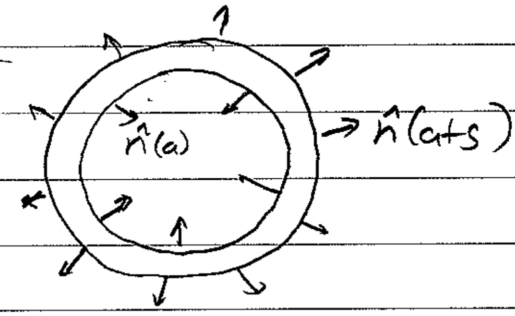


Image charge mimicking the effect of q and surface charge generating $\vec{E}$ for $z<0$

The bound surface charge at boundary of dielectric

$\sigma_b = \hat{n} \cdot \vec{P}$



$\Rightarrow \sigma_b(a) = -\left(\frac{\chi}{1+\chi}\right) \frac{Q}{4\pi a^2} \qquad \sigma_b(a+s) = +\left(\frac{\chi}{1+\chi}\right) \frac{Q}{4\pi (a+s)^2}$

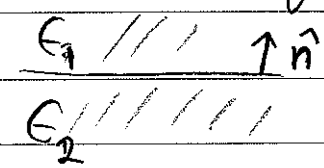
Note: The surface charge at $r=a$ is negative, attracted by positive Q. The dielectric acts as an "imperfect conductor", partially sheilding out the field ~~of~~ the charged sphere.

As $\chi \to \infty$ $\sigma_b(a) \to \frac{Q}{4\pi a^2}$, Looks like perfect conductor

Boundary conditions for dielectrics

Suppose there is no free charge anywhere

$\Rightarrow \vec{\nabla} \cdot \vec{D} = 0$



$\boxed{D_{\perp_1} - D_{\perp_2} = \frac{\sigma_{free}}{\epsilon_0}}$ Normal component of $\vec{D}$

If linear $\Rightarrow$ $\epsilon_1 E_{\perp_1} - \epsilon_2 E_{\perp_2} = \frac{\sigma_{free}}{\epsilon_0}$

Since $\vec{\nabla} \times \vec{E} = 0$ $\Rightarrow$ $E_{\parallel_1} = E_{\parallel_2}$

$\Rightarrow$ $z > 0$

$$V_> = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

$z < 0$

$$V_< = \frac{1}{4\pi\epsilon} \frac{q''}{\sqrt{x^2+y^2+(z-d^2)}}$$

B.C. (i)

$$\left.\frac{\partial V_>}{\partial x}\right|_{z=0} = \left.\frac{\partial V_<}{\partial x}\right|_{z=0} \Rightarrow \frac{1}{\epsilon_0}(q+q') = \frac{1}{\epsilon_1} q''$$

$$\epsilon_0 \left.\frac{\partial V_>}{\partial z}\right|_{z=0} = \epsilon \left.\frac{\partial V_<}{\partial x}\right|_{z=0} \Rightarrow q - q' = q''$$

$$\Rightarrow \quad q' = -\left[\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right] q$$

$$\boxed{\begin{aligned} q' &= -\left[\frac{K-1}{K+1}\right] q \\ q'' &= \left[\frac{2K}{K+1}\right] q \end{aligned}}$$

Note: In limit $K \to \infty$, looks like conductor

$$q' = -q$$