Physics 405

Lecture 21: Work and Energy in Dielectrics

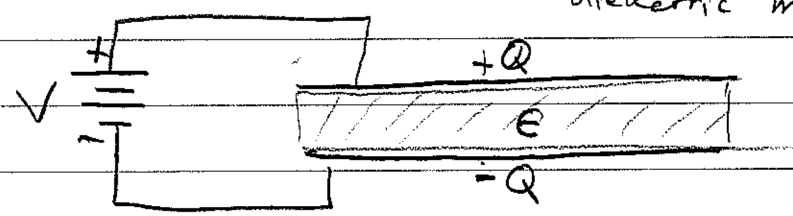
Capacitance

Recall the definition of capacitance:
- Given a potential difference V between two conductor, the induced surface charge / conductor

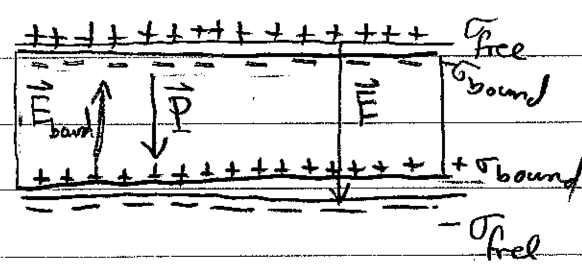
$$Q = CV$$

Suppose these conductors are in the presence of a dielectric material.

Canonical example: The parallel plate filled with dielectric material, permittivity ϵ



The induced surface bound charge will partially shield the the field of the surface free charge on the conductor. Thus, the battery will induce more free charge Q to establish the same potential difference V than it would in free space $\Rightarrow$ Capacitance increases



$$\vec{E} = \vec{E}_{free} + \vec{E}_{bound} = \vec{E}_{free} - \frac{\vec{P}}{\epsilon_0} \Rightarrow |\vec{E}| < |\vec{E}_{free}|$$

For linear dielectric, we can use Gauss's Law to determine the $\vec{E}$-field

$$\oint \vec{E} \cdot d\vec{a} = \frac{\rho_{free}}{\epsilon} \qquad \epsilon = \epsilon_0 (1 + \chi)$$

$\Rightarrow$ The total electric field inside the plates

$$|\vec{E}| = \frac{\sigma_{free}}{\epsilon} = \frac{Q}{A\epsilon}$$

$\therefore$ Potential difference $V = |\vec{E}| d = Q \frac{d}{A\epsilon} = \frac{Q}{C}$

$$\Rightarrow \boxed{C = \epsilon \frac{A}{d}}$$

Compare to in vacuum $\quad C = \epsilon_0 \frac{A}{d}$

$\Rightarrow$ Capacitance has increased by a factor of the dielectric constant K

Check:

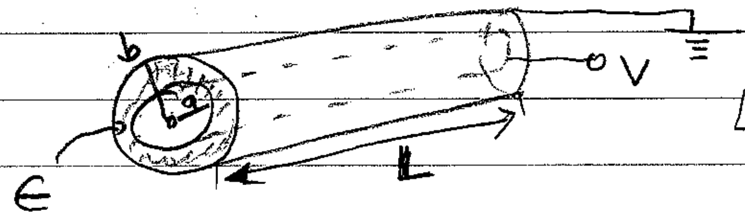
$$\vec{P} = \epsilon_0 \chi \vec{E}$$

$$\Rightarrow \vec{E} = \vec{E}_{free} - \chi \vec{E}$$

$$\Rightarrow \vec{E} = \frac{1}{1+\chi} \vec{E}_{free} = \frac{1}{K} \left(\frac{\sigma}{\epsilon_0} \right) \hat{z}$$

shielding

Example: Co-axial cable



$L \gg b$

Standary Gauss's Law $\Rightarrow$

$$E(r) = \begin{cases} 0 & r > b \\ \dfrac{Q/L}{2\pi\epsilon r} & a < r < b \\ 0 & r < a \end{cases}$$

$$\Rightarrow V_{ab} = \int_a^b E(r)\,dr = \frac{Q}{2\pi\epsilon L} \ln\left(\frac{b}{a}\right) = \frac{Q}{C}$$

$$\Rightarrow \boxed{C = 2\pi\epsilon L \frac{1}{\ln\left(\frac{b}{a}\right)}}$$

As in free space, only increased by K

Work and Energy

The work done by battery to establish charge distribution = Potential energy stored

$$U = \frac{1}{2} C V^2 = K\left(\frac{C_0 V^2}{2}\right) \quad \left(\text{For linear dielectric}\right)$$

But we found $\frac{1}{2} C_0 V^2 = \frac{\epsilon_0}{2} \int E^2 d^3r$

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Thus, in a linear dielectric,

$$\boxed{U = \frac{\epsilon}{2}\int |\vec{E}|^2 d^3r = \int \frac{\vec{D}\cdot\vec{E}}{2} d^3r}$$

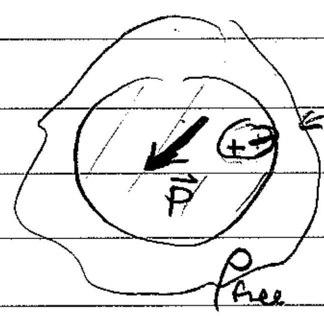
Example: Parallel Plates

$$U = \frac{\epsilon}{2}\int |\vec{E}|^2 d^3r = \frac{\epsilon}{2}|\vec{E}|^2 A d$$ (ignoring fringing fields)

$$U = \frac{\epsilon}{2}\left(\frac{\sigma}{\epsilon}\right)^2 A d = \frac{1}{2\epsilon}\frac{Q^2}{A} d = \frac{1}{2}\frac{Q^2}{C}$$

$$= \frac{1}{2} C V^2 \quad \checkmark$$

More General Picture



$\Delta\rho_{free}$ Work done to be q in from ground (at ∞)

$$\Delta W = \int (\Delta\rho_f)\, V(\vec{r})\, d^3r$$

↑ Increment of work done to increment free charge against force of all existing charges.

$V(\vec{r})$ ← total potential due to all charges

~~For linear dielectrics~~, An increment of free charge $\Delta\rho_f$ gives rise to increment in the electric displacement

$$\vec{\nabla}\cdot\left(\Delta\vec{D}\right) = \Delta\rho_{free}$$

$$\Rightarrow \Delta W = \int \vec{\nabla}\cdot(\Delta\vec{D})\, V\, d^3r$$

$$\underset{\text{integration by parts}}{=} \int \Delta\vec{D}\cdot(-\vec{\nabla}V)\, d^3r + \underbrace{\cancel{\oint (\Delta\vec{D})\, V\cdot d\vec{a}}}_{\text{Surface at } \infty} \to 0$$

$$\Rightarrow \boxed{\Delta W = \int(\Delta\vec{D})\cdot\vec{E}\, d^3r}$$

Generally $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$$\Rightarrow \Delta W = \epsilon_0\int(\Delta\vec{E})\cdot\vec{E}\, d^3r + \int(\Delta\vec{P})\cdot\vec{E}\, d^3r$$

$$\text{now } (\Delta\vec{E})\cdot\vec{E} = \frac{1}{2}\Delta(\vec{E}\cdot\vec{E})$$

$$\Rightarrow \Delta W = \underbrace{\Delta\left[\frac{\epsilon_0}{2}\int|\vec{E}|^2 d^3r\right]}_{\substack{\text{Increment of} \\ \text{work to move} \\ \Delta\rho \text{ again a given} \\ \text{fixed distribution } \rho_{total}}} + \underbrace{\int\Delta\vec{P}\cdot\vec{E}\, d^3r}_{\substack{\text{Increment of} \\ \text{work to establish} \\ \Delta\vec{P}}}$$

For linear dielectrics

$$\vec{D} = \epsilon \vec{E} \Rightarrow (\Delta \vec{D}) \cdot \vec{E} = \frac{1}{2} \Delta (\vec{D} \cdot \vec{E})$$

$$\Rightarrow \Delta W = \Delta \left[\frac{1}{2} \int \vec{D} \cdot \vec{E} \, d^3r \right]$$

$\Rightarrow$ Total work W = Potential energy stored in linear dielectric $\equiv U$

$$\Rightarrow \boxed{U = \frac{1}{2} \int \vec{D} \cdot \vec{E} = \frac{\epsilon}{2} \int |\vec{E}|^2 d^3r}$$