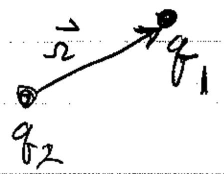
# Physics 405 : Lecture 22

## Introduction to Magnetostatics

Recall the definition of the electric field



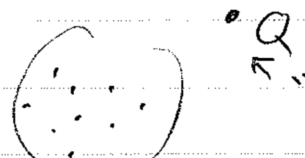

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{\mathcal{R}^2} \hat{\mathcal{R}} = \frac{1}{4\pi\epsilon_0} q_1 q_2 \left( \frac{\vec{\mathcal{R}}}{\mathcal{R}^3} \right)$$

Coulomb's Law : Action at a distance

$$\Rightarrow \quad \vec{F}_{12} = q_1 \vec{E}_2(\vec{r}_1)$$

local $\vec{E}$ @ $\vec{r}_1$ due to $q_2$

$$\vec{E}_2(\vec{r}_1) = \frac{1}{4\pi\epsilon_0} q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$




"test charge" @ $\vec{r}$

Collection of "external charges"

Static

$$\vec{F} = Q \vec{E}(\vec{r})$$

$$\vec{E} = \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3}$$

$$= \int d^3r' \frac{\rho(\vec{r}')}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

We took this empircal definition

and derived from in the general field eqns. of electro statics :

$$\boxed{\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0}, \quad \boxed{\vec{\nabla} \times \vec{E} = 0}$$

What happens when the charges move?

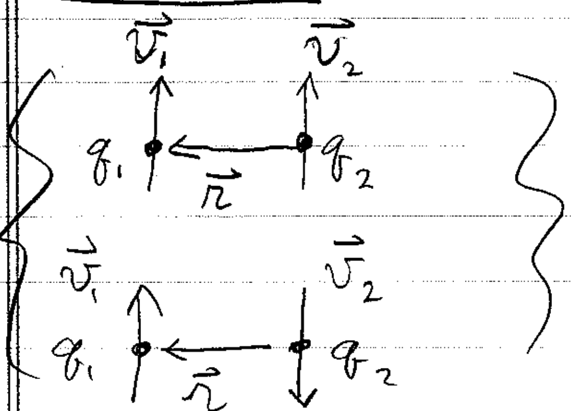
First empirical laws: Constant velocity, $v \lll c$ (speed of light)

Consider two point charges, $q_1, q_2$, separation vector $\vec{r}$ from $2 \to 1$, moving with velocities $\vec{v}_1$ and $\vec{v}_2$.

- Empiracle results: Addition force in addition to coulomb

- <u>Four cases</u>

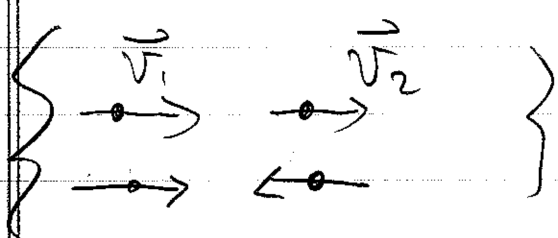
(i)



$$\vec{F}_{12} \sim \mp \hat{r} \frac{q_1 q_2}{r^2} v_1 v_2$$

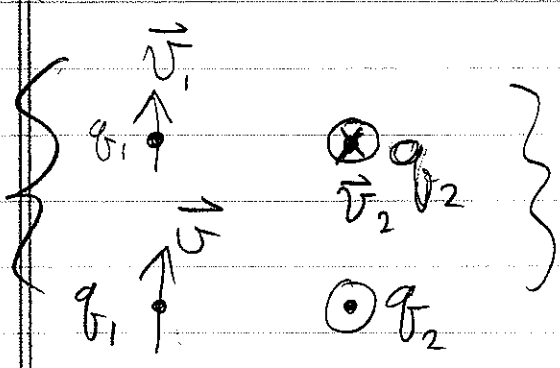
on 1 due to 2

(ii)



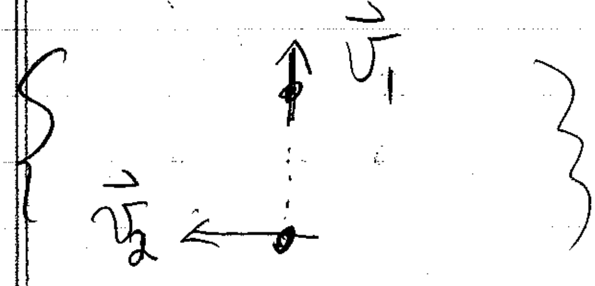
$$\vec{F}_{12} = 0$$

(iii)



$$\vec{F}_{12} = 0$$

(iv)



$$\vec{F}_{12} \sim \vec{v}_2 v_1 \frac{q_1 q_2}{r^2}$$

$$\vec{F}_{21} \sim 0$$

The additional force between charges is a *magnetic* interaction

$$\vec{F}^{\,magnetic}_{1,2} = k_m \frac{q_1 q_2}{r^2} \vec{v}_1 \times (\vec{v}_2 \times \hat{r}) \qquad \left(\text{non-relativistic}\right)$$

$k_m$ ⇑ Constant that determines the units

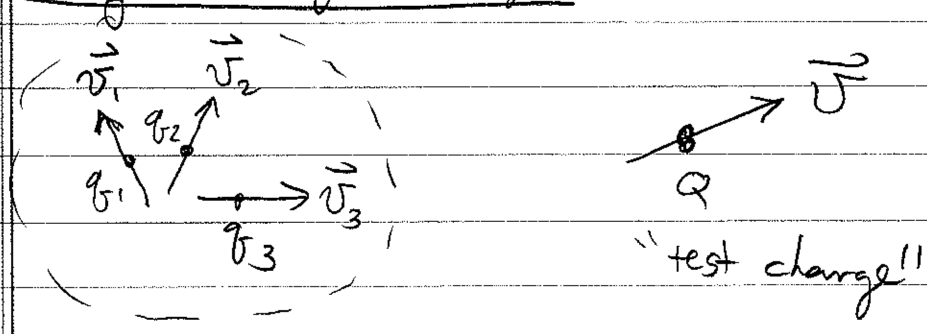
$$k_m = \frac{1}{4\pi \epsilon_0 c^2} \equiv \frac{\mu_0}{4\pi}$$

$$\frac{\mu_0}{4\pi} = 10^{-7} \frac{(\text{Newtons})(\text{sec})^2}{(\text{Coulomb})^2}$$

Note: $|\vec{F}^{\,mag}_{12}| \ll |\vec{F}^{\,electric}_{12}|$

when $\frac{v}{c} \ll 1$ non-relativistic

Many moving charges



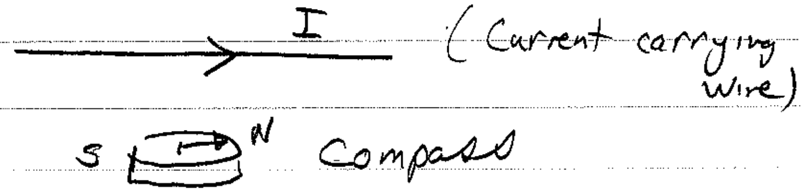

$$\vec{F}^{\,magnetic} = Q\vec{v} \times \vec{B}(\vec{r})$$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \sum_i q_i \vec{v}_i \times \frac{\hat{r}_i}{r_i^2}} \quad \text{Magnetic field}$$

## Conclusions

(1) Charges in motion produce a magnetic field
Current is the source of $\vec{B}$

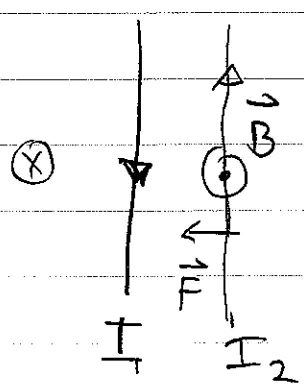
Oerstead 1819




(2) A Charge Q moving with velocity $\vec{v}$ feels a force

$$\vec{F}_{mag} = Q\,\vec{v} \times \vec{B}$$ : Lorentz Force Law


Anti-Parallel (parallel) current carrying wires attract (repell)

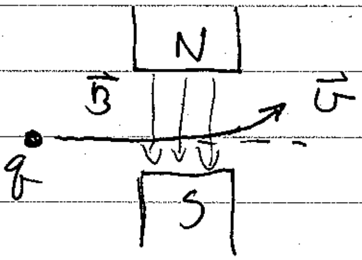



$$\text{Force}/\text{length} = \frac{\mu_0}{4\pi} \frac{I_1 I_2}{d}$$

## Motion of charge particles in $\vec{E} + \vec{B}$ fields

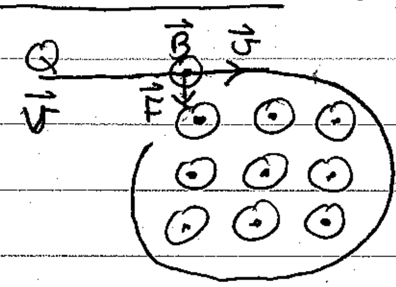
Lorentz force $\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$

(usually $\vec{E}$ dominates unless $|\vec{v}| \sim c$)




deflect charge

Cyclotron motion (Uniform $\vec{B}$ field




$\vec{F}_m = Q\vec{v} \times \vec{B}$

(Centripetal force)

Circular motion $|\vec{F}_m| = QvB = Mv^2/R$

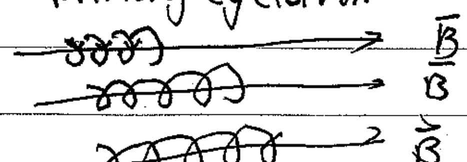
$$\Rightarrow R = \frac{Mv}{QB} \quad \text{(cyclotron radius)}$$

identify momentum

Angular frequency $\omega = \frac{v}{R} = \frac{QB}{M}$ (independent of $v$)

- Method of identify momenta of elementary particles
- Method of confining a plasma (gas of ions) for fusion

Drifting cyclotron motion



## Currents: The Source of $\vec{B}$-field

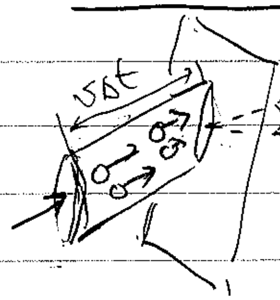
$I = \frac{\text{charge}}{\text{time}}$ (flow of charge)

Consider fluid-like picture:
Charge density $\rho$, moving with local velocity $\vec{v}$

Define <u>current density</u> $\vec{J} = \rho(\vec{r})\, v(\vec{r})$

$\Delta q = \rho(v\Delta t \Delta A \cos\theta)$

$\Rightarrow \frac{\Delta q}{\Delta t} = \underbrace{\rho \vec{v}}_{\vec{J}} \cdot \hat{n}\, \Delta A$



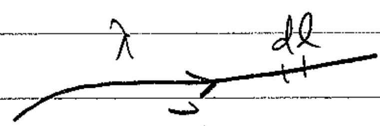

Flux of charge

$$I = \int_S \vec{J} \cdot d\vec{a}$$

$\vec{J}$ is the source of $\vec{B}$
as $\rho$ " " " of $\vec{E}$

## Current Distributions (steady velocity)

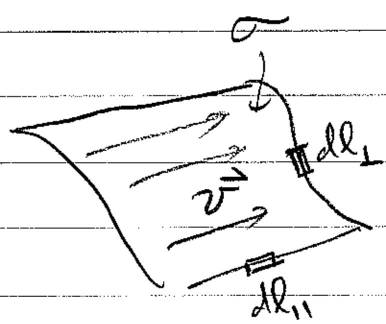
Line charge flow




$dq = \lambda\, dl$

$\Rightarrow I = \lambda \frac{dl}{dt} = \lambda v$

Surface charge flow



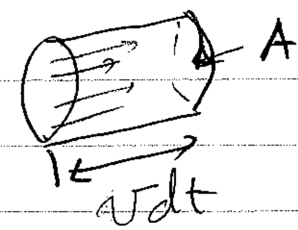

$dq = \sigma\, da = \sigma\, dl_\perp\, dl_\parallel$

$\Rightarrow I = \underbrace{(\sigma v)}_{K}\, dl_\perp$

$K$ ← surface current density

$[K] = \frac{\text{Current}}{\text{Length}}$

Volume charge density flow




$$dq = \rho v A\,dt \Rightarrow I = \rho v A$$

$$J = \rho v = \text{"Current density"}$$

$$[J] = \frac{\text{Current}}{\text{Area}}$$

<u>Force on continuous distribution</u>

$$\vec{F} = \int dq\,\vec{v} \times \vec{B}$$

Line charge $dq = \lambda\,dl \Rightarrow \vec{F} = \int I\,d\vec{l} \times \vec{B}$

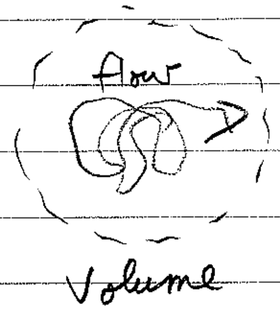
Surface charge $dq = \sigma\,da \Rightarrow \vec{F} = \int da\,\vec{K} \times \vec{B}$

Volume charge $dq = \rho\,d^3r \Rightarrow \vec{F} = \int d^3r\,\vec{J} \times \vec{B}$

<u>Key to source of $\vec{B}$</u>

$$\underset{\text{point}}{dq\,\vec{v}} \Longleftrightarrow \underset{\text{line}}{I\,d\vec{l}} \Longleftrightarrow \underset{\text{surface}}{\vec{K}\,da} \Longleftrightarrow \underset{\text{volume}}{\vec{J}\,d^3x}$$

## Conservation of Charge




Net flow out of volume

$$I_{out} = \oint_S \vec{J} \cdot d\vec{a}$$

$$= -\frac{d}{dt} Q_{enc}$$

decrease in enclosed charge

$$\Rightarrow \oint_S \vec{J} \cdot d\vec{a} = -\frac{d}{dt} \int_V d^3r \, \rho(\vec{r}) = -\int_V d^3r \frac{\partial \rho}{\partial t}$$

$$\parallel$$

$$\int (\vec{\nabla} \cdot \vec{J}) \, d^3r$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}}$$

"Magnetostatics" $\frac{\partial \rho}{\partial t} = 0$ : No local change in charge density

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = 0}$$ Magnetostatic $\vec{J}$