Physics 405: Lecture 23

Magneto-Static Fields

Last time we stated the empirical laws of magnetostatics

Lorentz Force Law: $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

Currents (charges in motion) are a source of the magnetic field.

Current density: $\vec{J}$ = Charge "flux density"

$= \dfrac{\text{Charge/time}}{\text{Area}}$

$$\int_S \vec{J}\cdot d\vec{a} = \text{Current flow through surface } S$$

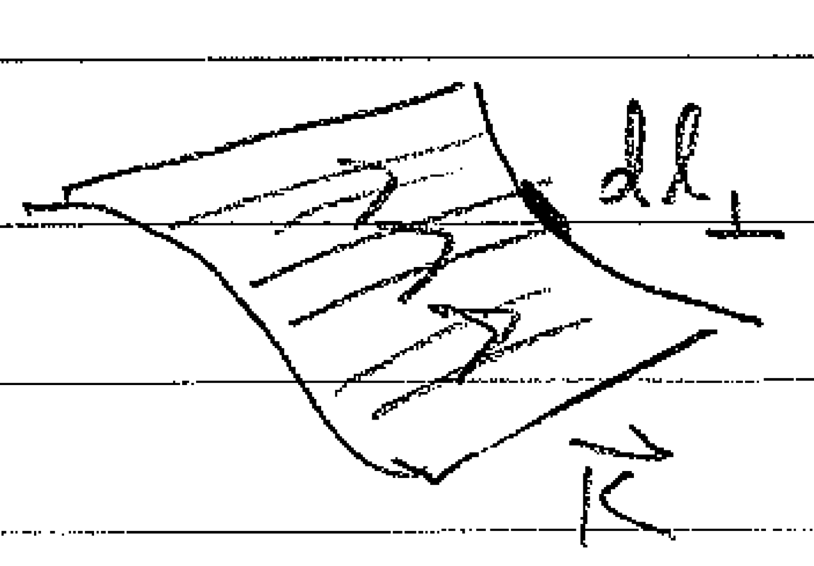
Charge distributed in volume, locally moving @ velocity $\vec{v}$

$$\vec{J} = \rho\vec{v}$$

Charge distributed on surface, locally moving @ velocity $\vec{v}$

$\vec{K} = \sigma\vec{v}$ (surface current length)

$= \dfrac{\text{charge/time}}{\text{length}}$



$I = \int K\, dl_\perp$

Conservation of charge:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \qquad \text{"Continuity Equation"}$$

<u>Magnetostatics</u>: Steady current $\rightarrow \frac{\partial \rho}{\partial t} = 0$

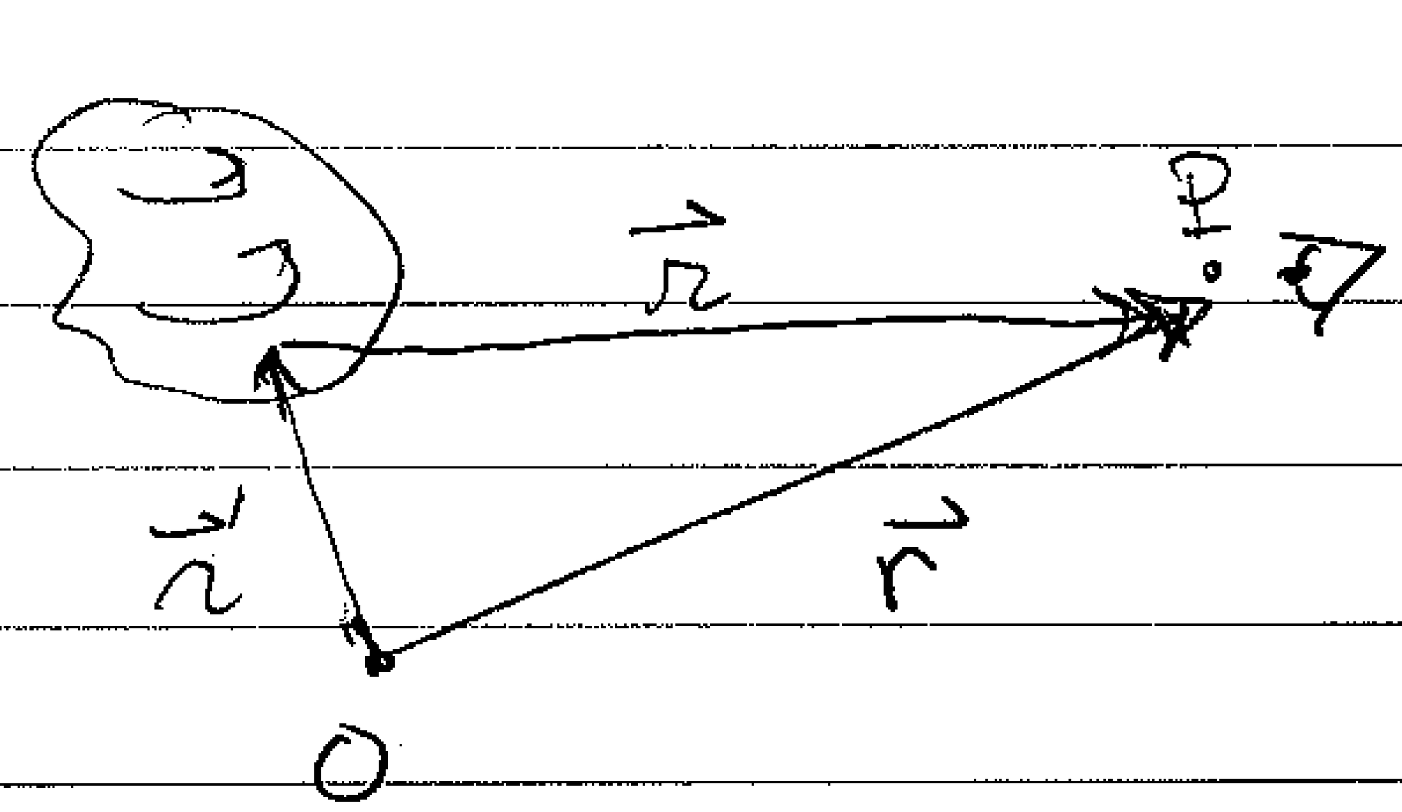
$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = 0}$$

(Magnetostatics as consistent with electrostatics $\frac{\partial \rho}{\partial t} = 0$)

<u>Biot-Savart</u>

Given a magnetostatic current density

$\vec{J}$ or $\vec{K}$ or I



Biot-Savart law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \; \vec{J}(\vec{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0}{4\pi}\int d^2a' \; \vec{K}(\vec{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

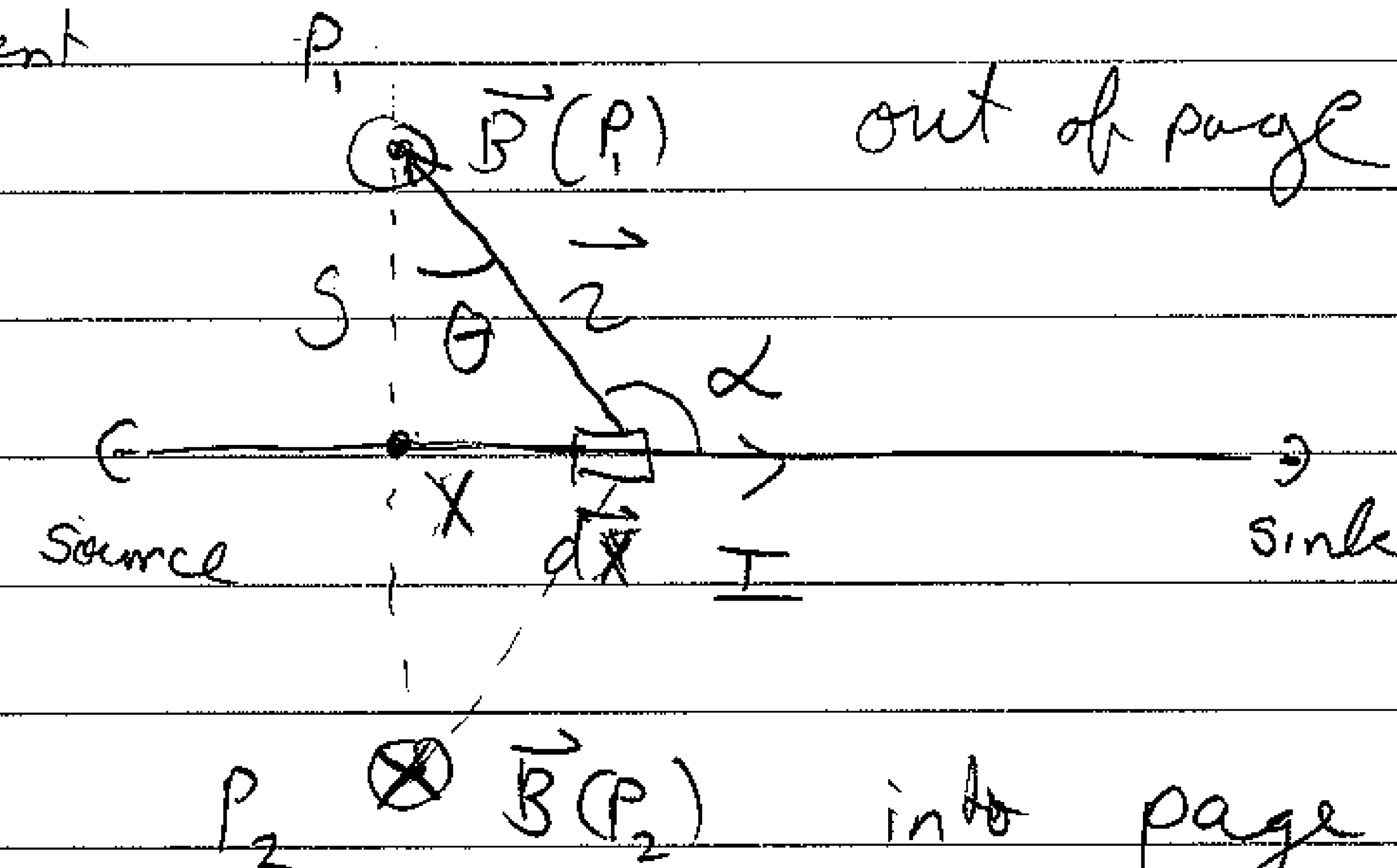
$$= \frac{\mu_0}{4\pi}\int I \, d\vec{l} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Note: A moving point charge is not a magnetostatic configuration since the local Charge density is changing. Nonetheless, when velocity is non-relativistic, $v << c$, we have, to good approximation the $\vec{B}$ field of q moving @ $\vec{v}$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} q\vec{v} \times \frac{\hat{\imath}}{\imath^2}$$

Magnetic field of a current carrying wire

The most basic of magneto-static fields (analogous to the $\vec{E}$-field of a Point charge) is $\vec{B}$ from a infinitely long straight wire with current I. Consider first a finite segment



$$d\vec{l} \times \hat{\imath} = |d\vec{x}|\,|\hat{\imath}| \sin\alpha = dx \cos\theta$$

let θ be the variable that parameterizes the position of current element

$$\tan\theta = \frac{X}{S} \Rightarrow \sec^2\theta\, d\theta = \frac{dX}{S}$$

$$\Rightarrow dx = s \sec^2\theta \, d\theta = \frac{s \, d\theta}{\cos^2\theta}$$

$$\Rightarrow \left|\vec{B}(P)\right| = \frac{\mu_0 I}{4\pi} \int_{x_1}^{x_2} (dx \cos\theta) \frac{1}{r^2}$$

$$= \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{s \, d\theta}{\cos^2\theta} \cos\theta \frac{1}{r^2}$$

Aside $\frac{s}{r} = \cos\theta \Rightarrow \frac{1}{r^2} = \frac{\cos^2\theta}{s^2}$

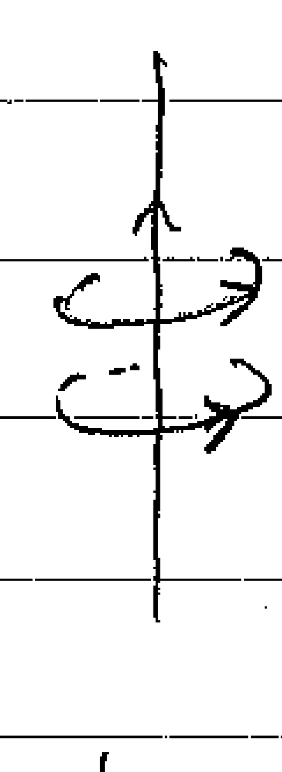
$$\Rightarrow \left|\vec{B}(P)\right| = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} d\theta \cos\theta$$

$$\boxed{\vec{B}(P) = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1)}$$

Now take limit as length $\to \infty$

$$\theta_1 \to -\frac{\pi}{2}, \quad \theta_2 \to \frac{\pi}{2}$$

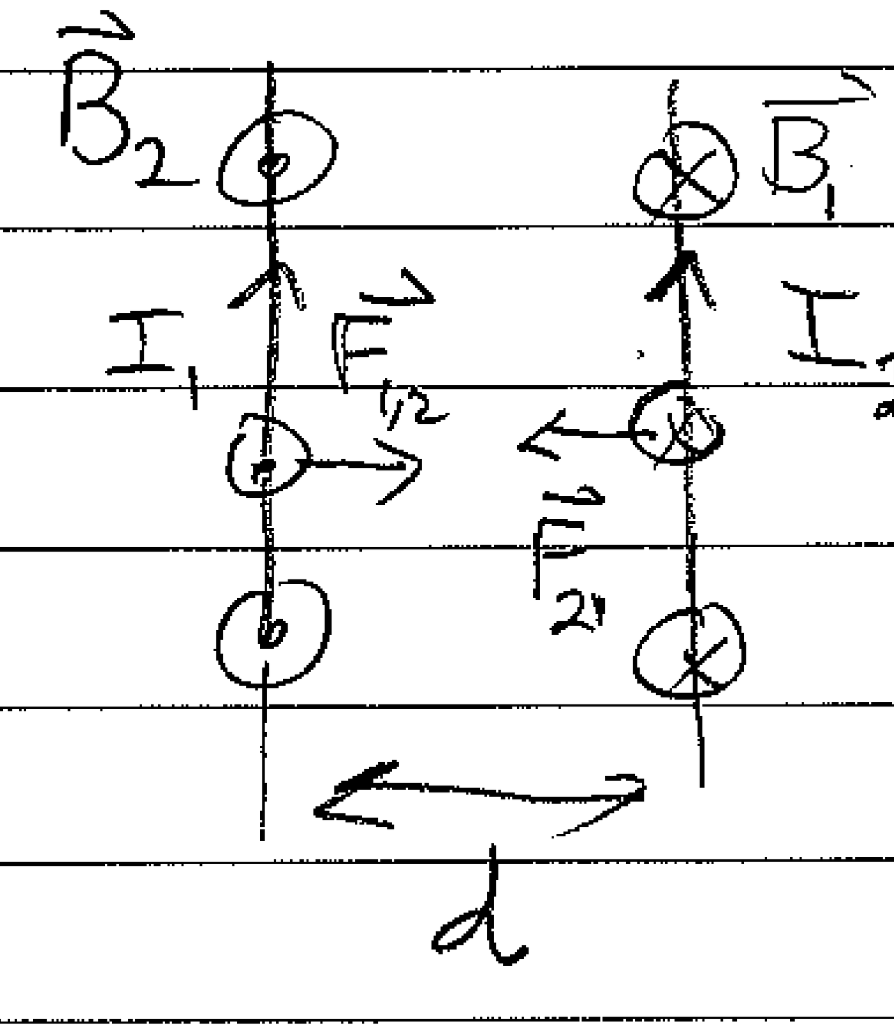
$$\Rightarrow \boxed{\vec{B}(P) = \frac{\mu_0 I}{2\pi s} \hat{\phi}}$$



Right-hand rule

Magnetic field around infinite wire $\sim \frac{1}{s}$
where s is distance $\perp$ to wire

Force between parallel wires



$$\vec{F}_{1,2} = \int I_1 \, d\vec{l}_1 \times \vec{B}_2$$

$$|\vec{F}_{12}| = I_1 B_2 \int dl_1$$

$$= \frac{\mu_0}{4\pi} \frac{I_1 I_2}{d} \int dl_1$$

$$= |\vec{F}_{21}|$$

Force between two (infinitely long) parallel wires, per unit length

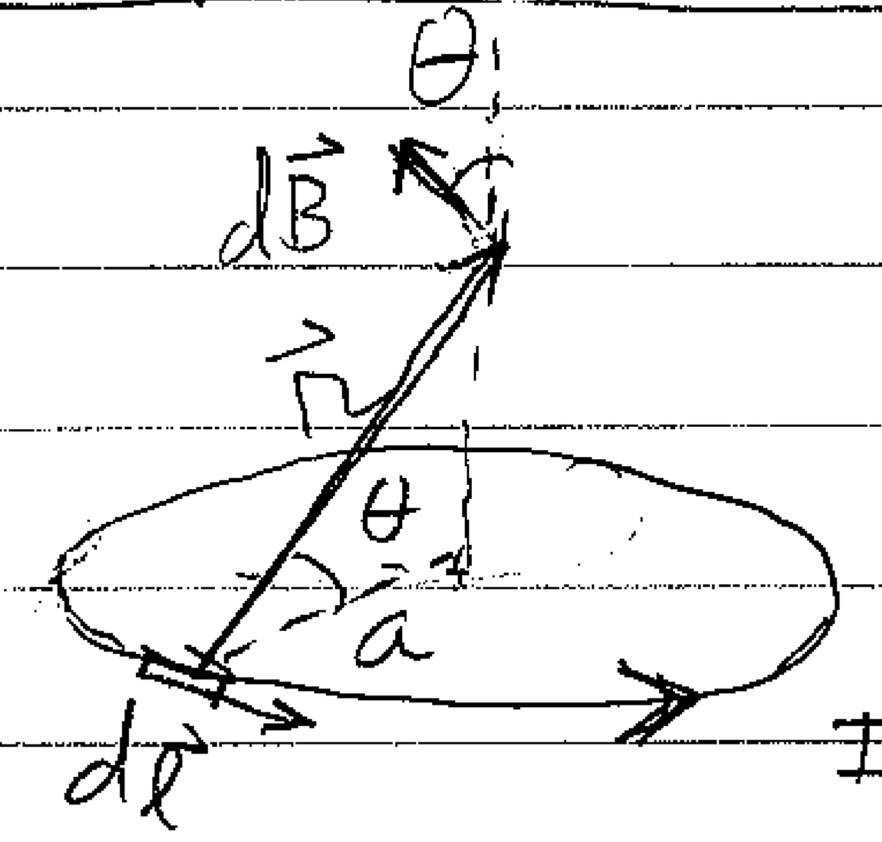
$$f = \frac{\mu_0 I_1 I_2}{4\pi d}$$

- Attract for current in same direction
- Repulsive for current in opposite directions

Note; Symmetry considerations.
In electrostatics, given an (infinite) line charge, we argued that $|\vec{E}|$ was naturally described in cylindrical coordinates, ρ, ϕ, z. We argued that $|\vec{E}|$ could only depend on ρ and point in the $\hat{\rho}$ direction. For $\vec{B}$, $|\vec{B}|$ also only depends of ρ (as it must from symmetry). However, the direction of $\vec{B}$ is $\hat{\phi}$ which varying with ϕ: $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$. This is allowed since the current has a direction which ~~also~~ gives handedness.

B-field of a current loop

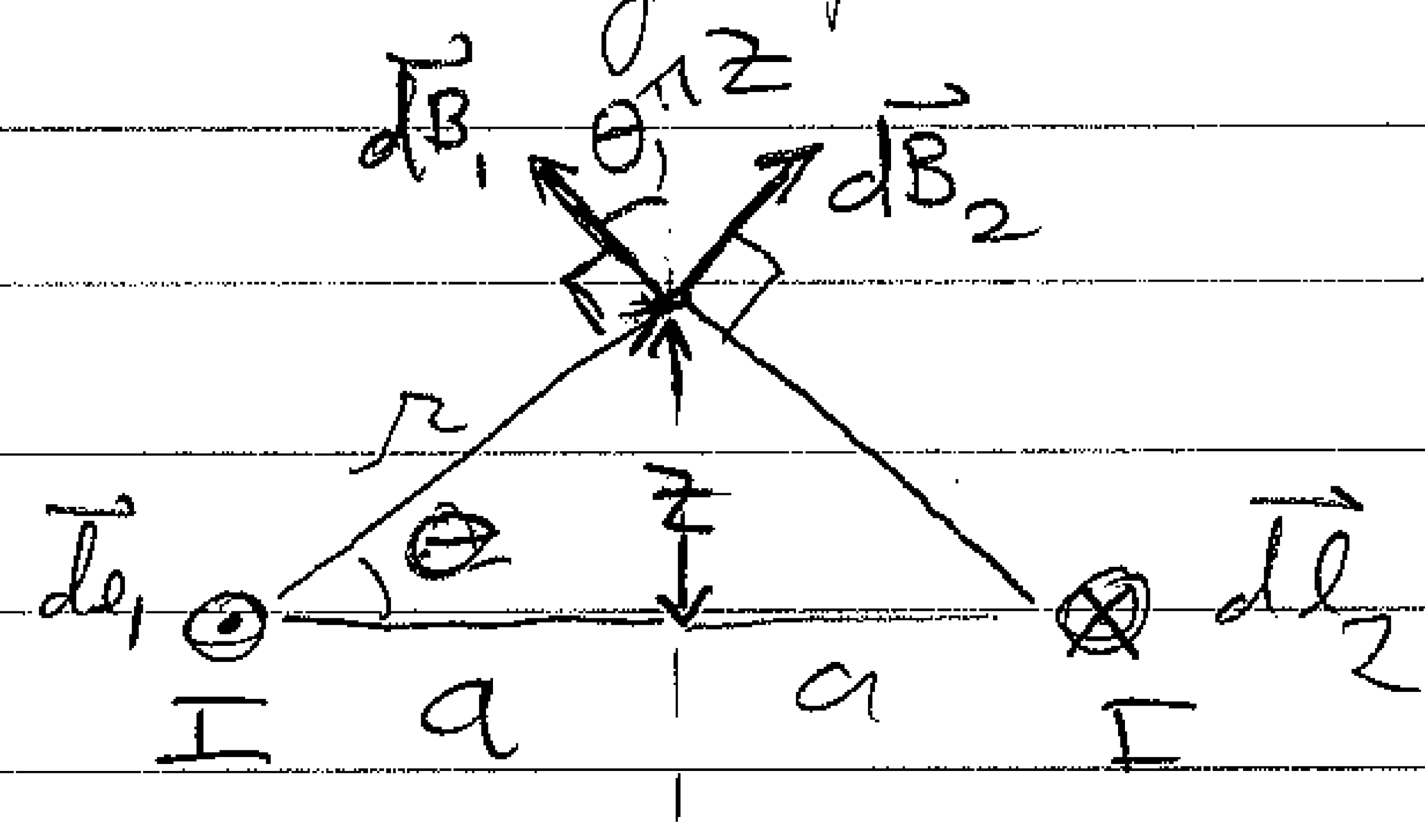
Let us find $\vec{B}$ on the z-axis of symmetry



$$\vec{B}(\cdot) = \int d\vec{B}$$

$$d\vec{B} = \frac{\mu_0}{4\pi} I\, d\vec{l} \times \frac{\hat{r}}{r}$$

Look in any plane containing z-axis



$$|d\vec{B}_1| = |d\vec{B}_2| = \frac{\mu_0}{4\pi} I\, dl \frac{1}{r^2} \quad (d\vec{l} \perp \hat{r})$$

Only the z-component survives when we superpose the contributions from opposite ends of the loop

$$dB_z = |d\vec{B}| \cos\theta = I\, dl \frac{\cos\theta}{r^2}$$

$$= \frac{\mu_0}{4\pi} I\, dl \frac{a}{r^3} = \frac{\mu_0}{4\pi} I\, dl \frac{a}{(\sqrt{a^2+z^2})^3}$$

$$\Rightarrow \vec{B}(z) = B_z(z)\,\hat{z} = \oint d\vec{B}_z = \frac{\mu_0}{4\pi} \frac{I a}{(a^2+z^2)^{3/2}} \oint dl$$

Thus, since $\oint dl = 2\pi a$

$$\boxed{\vec{B}(z) = \frac{\mu_0}{4\pi}\left[\frac{2\pi a^2 I}{(a^2+z^2)^{3/2}}\right]\hat{z}}$$

Note: when $z >> a$

$$\vec{B}(z) \approx \frac{\mu_0}{4\pi}\left[\frac{2\pi a^2 I}{z^3}\right]\hat{z}$$

Field falls off a $\frac{1}{z^3}$ $\Longrightarrow$ Dipole field!

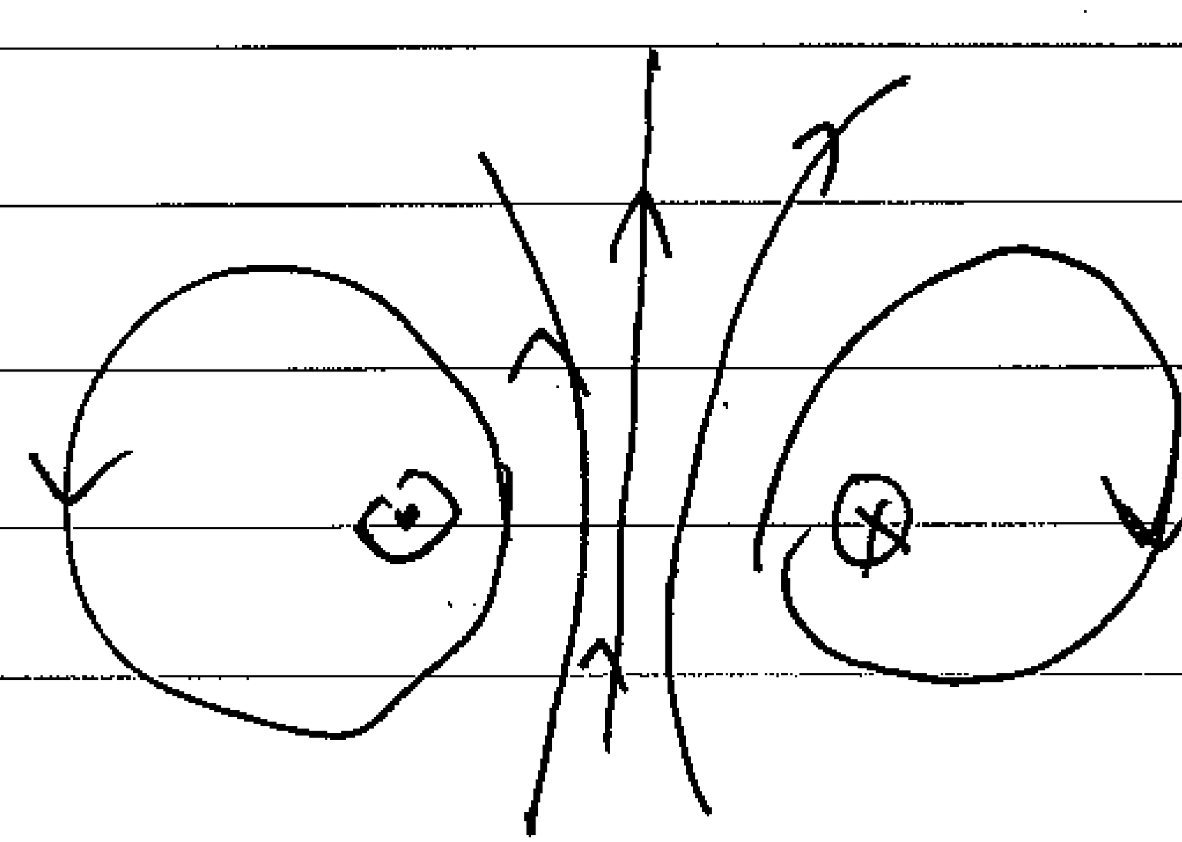
Recal for electric dipole $\vec{p} = p\hat{z}$, on axis of dipole

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\left[\frac{2p}{z^3}\right]\hat{z}$$

$\uparrow p$

Current loop $\Longrightarrow$ magnetic dipole

$|\vec{m}| = (\pi a^2) I = (\text{Area}) I$ as we will see



Physical magnetic dipole