Physics 405  Lecture 23

Magnetic-Static Fields

Last time we stated the empirical laws of magnetisms:

Lorentz Force Law: \( \vec{F} = q \vec{E} + q \vec{v} \times \vec{B} \)

Currents (charge in motion) are a source of the magnetic field.

Current density: \( \vec{J} = \text{Charge "flux density"} \)
\[
\vec{J} = \frac{\text{charge}}{\text{Area}} \cdot \text{time}
\]

\[
\int_{\Sigma} \vec{J} \cdot d\sigma = \text{Current flow through surface } \Sigma
\]

Charge distributed in volume, locally moving @ velocity \( \vec{v} \)

\( \vec{J} = \rho \vec{v} \)

Charge distributed on surface, locally moving @ velocity \( \vec{v} \)

\( \vec{K} = \sigma \vec{v} \) (surface current intensity)
\[
\vec{K} = \frac{\text{current}}{\text{length}}
\]

\( \int_{\Sigma} \vec{K} \cdot d\Sigma \)

\( \int_{\Sigma} \vec{K} \cdot d\Sigma \)
Conservation of charge:

\[ \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad "\text{Continuity Equation}" \]

Magnetostatics: Steady current \( \Rightarrow \frac{\partial \vec{B}}{\partial t} = 0 \)

\[ \Rightarrow \nabla \cdot \vec{J} = 0 \]

(Magnetostatics as consistent with electrostatic \( \frac{\partial \vec{E}}{\partial t} = 0 \))

**Biot-Savart**

Given a magnetostatic current density \( \vec{J} \) or \( \vec{K} \) or \( \vec{I} \)

**Biot-Savart law**

\[ \vec{B}(r) = \frac{\mu_0}{4\pi} \int d^3r' \vec{J}(r') \times \frac{\hat{r}}{r^2} \]

\[ = \frac{\mu_0}{4\pi} \int d^3r' \vec{K}(r') \times \frac{\hat{r}}{r^2} \]

\[ = \frac{\mu_0}{4\pi} \int I \, dl \times \hat{r} \]
Note: A moving point charge is not a magneto-static configuration since the local charge density is changing. Nonetheless, when velocity is non-relativistic, \( v \ll c \), we have, to good approximation, the \( \vec{B} \) field of a moving \( q \) moving at \( \vec{v} \):

\[
\vec{B}(\vec{r}) = \frac{\mu_0 q \vec{v} \times \hat{\vec{r}}}{4\pi r^2}
\]

**Magnetic field of a current-carrying wire**

The most basic of magneto-static fields (analogous to the \( \vec{E} \)-field of a point charge) is \( \vec{B} \) from a infinitely long straight wire with current \( I \).

Consider first a small segment \( p \) of the wire.

\[p \times \vec{B}(p)\text{ out of page}\]

\[\vec{S} \text{ out of page}\]

\[\vec{E} \text{ out of page}\]

\[\vec{C} \text{ out of page}\]

\[\text{Source}\]

\[\text{Sink}\]

\[\vec{p}_2 \times \vec{B}(p_2) \text{ into page}\]

\[d\vec{l} \times \hat{n} = l dx |\hat{n}| \sin \alpha = dx \cos \theta\]

Let \( \theta \) be the variable that parameterizes the position of current element.

\[\tan \theta = \frac{d}{s} \Rightarrow \sec^2 \theta d\theta = \frac{dx}{s}\]
\[ dx = s \sec^2 \theta \, d\theta = \frac{s \, d\theta}{\cos^2 \theta} \]

\[ |\mathbf{B}(\mathbf{p})| = \frac{M_0 I}{4\pi} \int_{x_1}^{x_2} (dx \cdot \cos \theta) \frac{1}{r^2} \]

\[ = \frac{M_0 I}{4\pi} \int_{\theta_1}^{\theta_2} s \, d\theta \frac{1}{\cos \theta} \frac{1}{r_2} \]

Aside \hspace{1cm} \frac{s}{r} = \cos \theta \hspace{1cm} \frac{1}{r^2} = \frac{\cos^2 \theta}{s^2} \]

\[ |\mathbf{B}(\mathbf{p})| = \frac{M_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos \theta} \cos \theta \]

\[ \mathbf{B}(\mathbf{p}) = \frac{M_0 I}{4\pi s} \left( \sin \theta_2 - \sin \theta_1 \right) \]

Now take limit as length \( \to \infty \)

\( \theta_1 \to -\frac{\pi}{2}, \quad \theta_2 \to \frac{\pi}{2} \)

\[ \mathbf{B}(\theta) = \frac{M_0 I}{2\pi s} \frac{\phi}{R} \]

Right-hand rule

Magnetic field around infinite wire \( \sim \frac{1}{s} \)

where \( s \) is distance to wire
Force between parallel wires

\[ F_{12} = \int I_1 \, dl_1 \times \vec{B}_2 \]

\[ |F_{12}| = \frac{M_0 I_1 I_2}{4\pi d} \int dl_1 \]

Force between two (infinitely long) parallel wires, per unit length

\[ f = \frac{M_0 I_1 I_2}{4\pi d} \]

- Attract for current in same direction
- Repel each other for current in opposite directions

Note: symmetry considerations.

In electrostatics, given an infinite line charge, we argued that \( |E| \) was naturally described in cylindrical coordinates, \( \rho, \phi, z \). We argued that \( |E| \) could only depend on \( \rho \) and point in the \( \phi \) direction. For \( \vec{B} \), \( |B| \) also only depends on \( \phi \) (as in must from symmetry). However, the direction of \( \vec{B} \) is \( \phi \) and \( \Phi \) varying with \( \phi \): \( \Phi = -\sin \phi \chi + \cos \phi \). This is allowed since the current has a direction which gives handedness.
B-field of a current loop

Let us find \( \vec{B} \) on the z-axis of symmetry \( \theta \), \( \phi \), \( r \), \( \vec{a} \).

\[
\vec{B}(\vec{r}) = \int d\vec{B}
\]

\[
d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{n}}{r^2}
\]

Look in any plane containing z-axis

\[
|d\vec{B}_1| = |d\vec{B}_2| = \frac{\mu_0}{4\pi} \frac{I d\vec{l}}{r^2}
\]

Only the z-component survives when we superpose the contributions from opposite ends of the loop

\[
d\vec{B}_z = |d\vec{B}| \cos \theta = \frac{I d\vec{l} \cos \theta}{r^2}
\]

\[
= \frac{\mu_0}{4\pi} \frac{I d\vec{l}}{r^3} = \frac{\mu_0}{4\pi} \frac{I d\vec{l}}{(\sqrt{a^2 + z^2})^3}
\]

\[
\Rightarrow \vec{B}(z) = \int_{\frac{z}{2}}^{\frac{z}{2}} d\vec{B}_z = \frac{\mu_0}{4\pi} \frac{I q}{(a^2 + z^2)^{3/2}}
\]
Thus, since \( \mathbf{B}_{\text{field}} = 2\pi q \)

\[
\mathbf{B}(z) \approx \frac{\mu_0}{4\pi} \left[ \frac{2\pi a^2 I}{z^3} \right] \hat{z}
\]

Field falls off as \( \frac{1}{z^3} \) \Rightarrow Dipole field.

Recall for electric dipole \( \mathbf{p} = p \hat{z} \),
on axis of dipole

\[
\hat{E}(z) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{2p}{z^3} \right] \hat{z}
\]

\( \hat{p} \)

Current loop \( \Rightarrow \) magnetic dipole

\(|\mathbf{m}| = (2\pi a^2) I = (\text{Area}) I \) as we will see

\[ \text{Physical} \]

\[ \text{magnetic dipole} \]