Properties of the Magneto-Static Field

Last time we saw that for a current loop, the magnetic field far from the loop was a dipole field.

On axis:

\[ \mathbf{B}(z) = \frac{\mu_0 I}{4\pi} \frac{2r}{z^3} \]

This is quite unlike the electric field far from a localized charge distribution, which generically is a monopole (unless the charge distribution is neutral).

The fact that the lowest nonvanishing contribution to magnetic fields is dipole is general. There are no magnetic monopoles, as we will see.

Another example: Rotating disk of charge, angular velocity \( \omega \).

Find \( \mathbf{B}(z) \)

\[ \mathbf{K} = \sigma \mathbf{J} \]

\[ \mathbf{J}(\mathbf{r}) = \sigma \mathbf{v}(\mathbf{r}) \]

\[ \mathbf{v}(\mathbf{r}) = \hat{\mathbf{w}} \times \mathbf{r} \]

\[ \Rightarrow \mathbf{K} = \sigma \hat{\mathbf{w}} r \phi \]

(Surface current density)
Biot-Savart Law

\[ \vec{B}(z) = \frac{\mu_0}{2\pi} \int \frac{\vec{I} \times \hat{n}}{r^2} \, d\sigma \]

The easiest approach is to break up the disk into rings, and use the solution we found for it:

Current in differential ring

\[ dI = K dl = K dr \]

\[ = \sigma \pi r^2 dr \]

\[ \implies \text{On axis, this differential contribution contributes to } dB \]

\[ dB(z) = \frac{\mu_0}{2} \frac{r^2 dI(r)}{(r^2 + z^2)^{3/2}} = \frac{\mu_0 \sigma \pi}{2} \frac{r^3 dr}{(r^2 + z^2)^{3/2}} \]

\[ \implies B(z) = \frac{\mu_0 \sigma \pi}{2} \int_0^R \frac{r^3 dr}{(r^2 + z^2)^{3/2}} \]

\[ B(z) = \frac{\mu_0 \sigma \pi}{2} \left[ -2z + \frac{R^2 + 2z^2}{\sqrt{R^2 + z^2}} \right] \]

Look at limit \( z \gg R \):

\[ B(z) \approx \frac{\mu_0 \sigma \pi}{2} \frac{R^2}{2} \left[ -2 + \frac{z^2 + R^2}{(R^2)^{3/2}} \right] \]

\[ \approx \frac{\mu_0 \sigma \pi}{2} \frac{R^2}{2} \left[ -2 + (2 + \frac{R^2}{z^2}) \left( 1 - \frac{1}{2} \frac{R^2}{z^2} \right) \right] \]

\[ B(z) \propto \frac{\mu_0 \sigma \pi R^4}{2z^3} \sim \frac{1}{z^3} \text{ dipolar} \]
Divergence and Curl of $\mathbf{B}$

Let us begin with the Biot–Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \mathbf{r}}{\mathbf{r}^2} \, d^3 \mathbf{r'}$$

$$\Rightarrow \nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int d^3 \mathbf{r'} \cdot \nabla \cdot \left( \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}}{\mathbf{r}^2} \right)$$

recall: $\mathbf{r} = \mathbf{r} - \mathbf{r}'$, $\nabla$ as on unprimed $\mathbf{r}$

Using product rule: $\nabla \cdot (\mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}}{\mathbf{r}^2}) = -\mathbf{J}(\mathbf{r}') \cdot \nabla \times \frac{\mathbf{r}}{\mathbf{r}^2}$

But we know $\nabla \times \frac{\mathbf{r}}{\mathbf{r}^2} = 0$ since $\frac{\mathbf{r}}{\mathbf{r}^2}$ is the electrostatic field of a unit point charge

$$\Rightarrow \left[ \nabla \cdot \mathbf{B} = 0 \right]$$

This is the mathematical statement of "there are no magnetic monopoles". The magnetic fields never converge or diverge from a point.

Aside: This law is empirical rather than fundamental. If some day one were to discover "magnetic charge" we would have to modify this law.
\\[
\vec{D} \times \vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{D} \times (\vec{J}(\vec{r}')) \times \hat{r}}{r'^2} \, d^3r',
\]

Using product rule:
\[
\vec{D} \times (\vec{J}(\vec{r}') \times \hat{r}^2) = \vec{J}(\vec{r}') \left( \frac{\vec{D} \cdot \hat{r}}{r'^2} \right)
\]

Now this part is tricky:
\[
\frac{\hat{r}}{r^2} = 4\pi \varepsilon_0 \text{ electric field of a unit charge } Q \text{ at } \vec{r}', \text{ observed at } \vec{r}
\]

\[
\Rightarrow \quad \vec{D} \cdot \frac{\hat{r}}{r^2} = \frac{4\pi \epsilon_0}{r^2} \rho_{\text{obs}}(\vec{r})
\]

Point at \( \vec{r}' \)

"Dirac delta function"

\[
\Rightarrow \quad \vec{D} \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^{(3)}(\vec{r} - \vec{r}')
\]

\[
\Rightarrow \quad \vec{D} \times \vec{B} = \mu_0 \int d^3 r' \, \vec{J}(\vec{r}') \delta^{(3)}(\vec{r} - \vec{r}')
\]

\[
\Rightarrow \quad \boxed{\vec{D} \times \vec{B} = \mu_0 \int \vec{J}' \quad \text{Amperé’s Law}}
\]