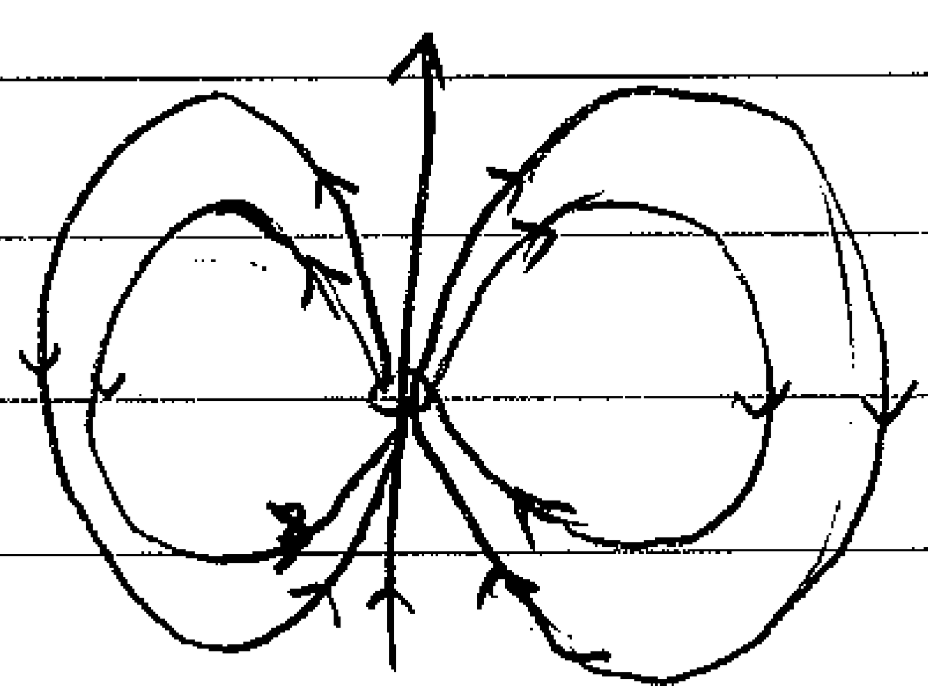
Physics 405: Lecture 24

Properties of the Magneto-Static Field

Last time we saw that for a current loop, the magnetic field far from the loop was a <u>dipole field</u>



On axis:

$$\vec{B}(z) = \frac{\mu_0}{4\pi}\,\frac{2\vec{m}}{z^3}$$

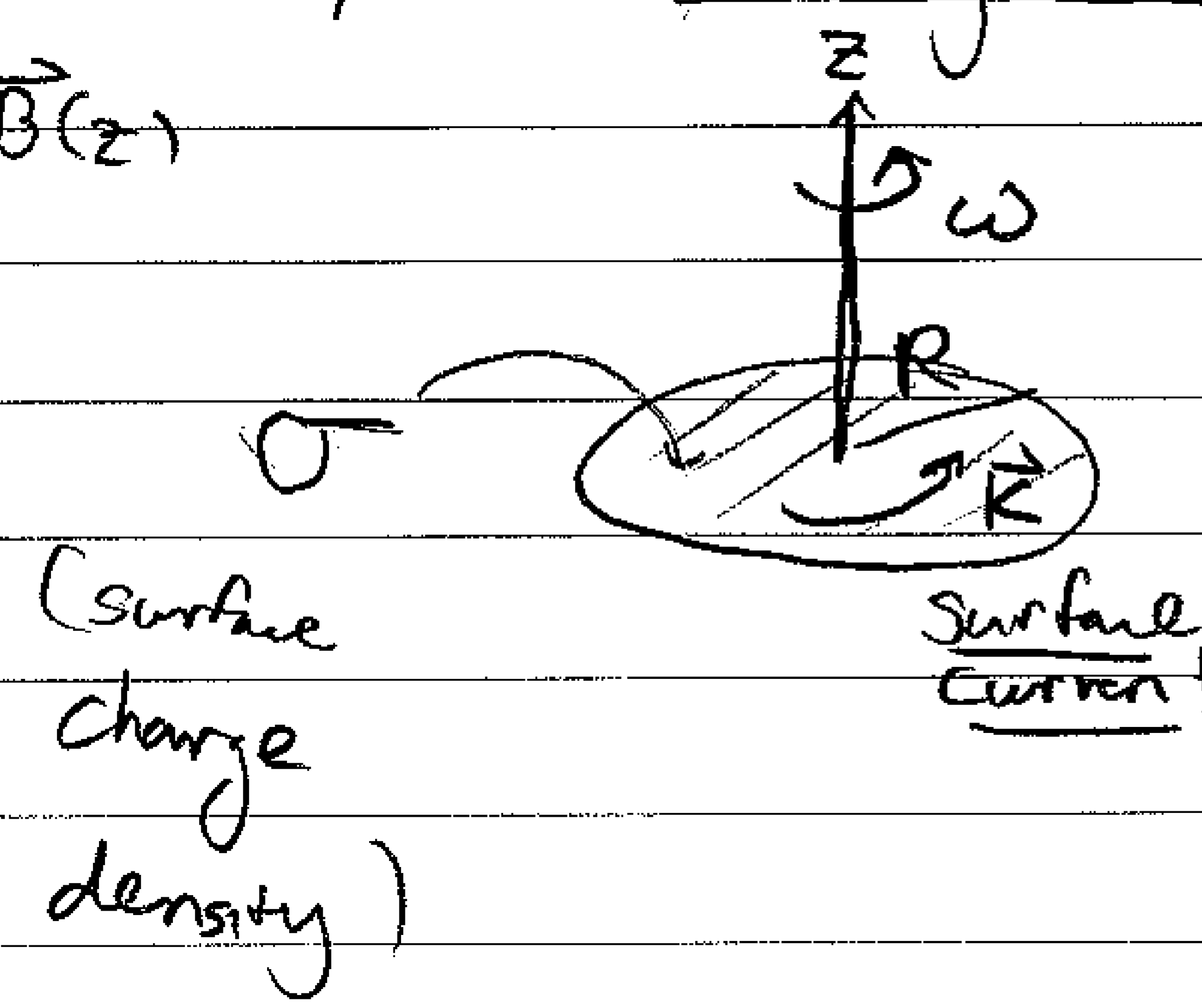
$$\vec{m} = I\,\underset{\sim}{A} \quad \text{loop area}$$

This is quite unlike the Electric field far from a localized charge distribution which generically is <u>monopole</u> (unless the charge distribution is neutral).

The fact that the lowest nonvanishing contribution to magnetic fields is dipole is generical, there are <u>no</u> <u>magnetic</u> <u>monopoles</u>, as we will see.

Another example: <u>Rotating disk</u> of charge, angular velocity ω

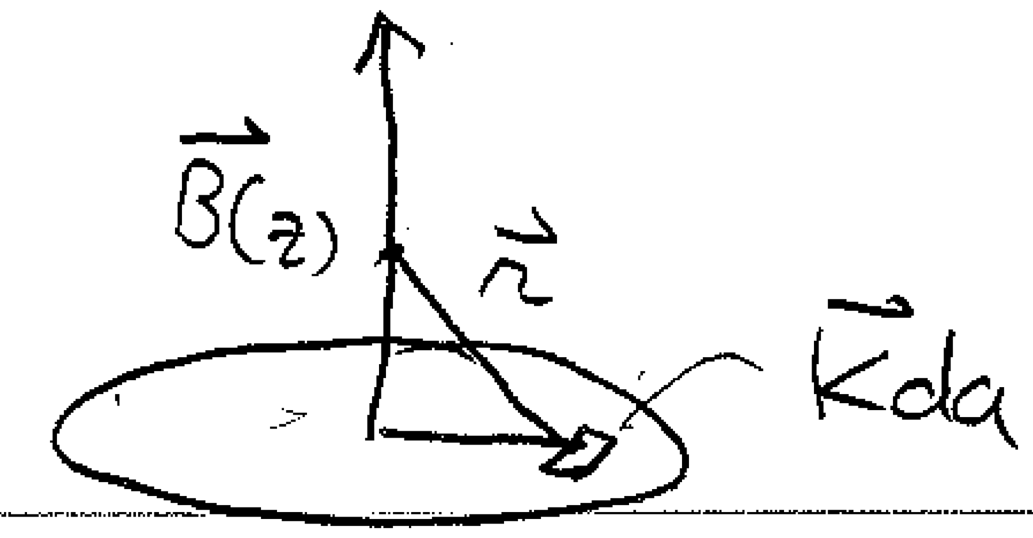
Find $\vec{B}(z)$



$$\vec{K}(\vec{r}) = \sigma\,\vec{v}(\vec{r})$$

$$\vec{v}(\vec{r}) = \vec{\omega}\times\vec{r} = \omega r\,\hat{\phi}$$

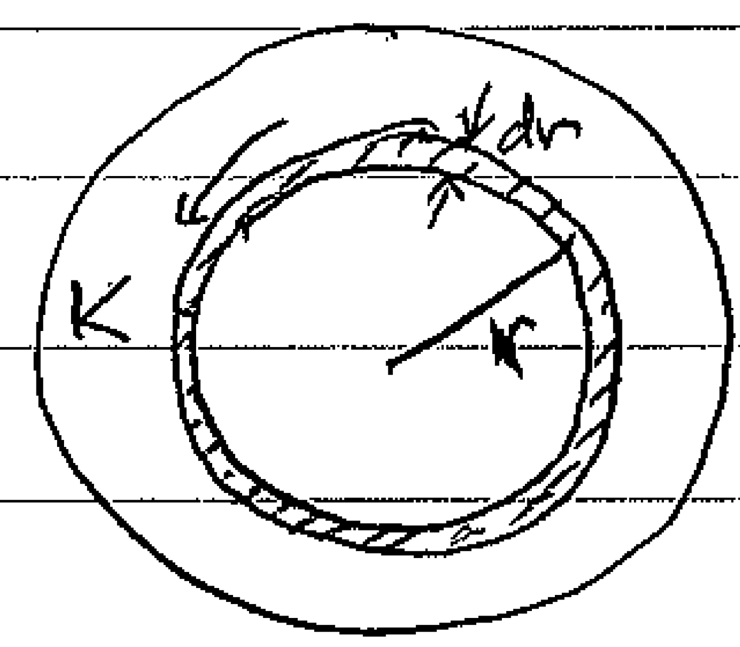
$$\Rightarrow \vec{K} = \sigma\omega r\,\hat{\phi}$$



Biot-Savart law

$$\vec{B}(z) = \frac{\mu_0}{4\pi} \int_{disk} da \, \vec{K} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

The easiest approach is to break up the disk into rings, and use the solution we found for it:



Current in differential ring

$$dI = K\, dl_{\perp} = K\, dr$$
$$= \sigma \omega r\, dr$$

$\Rightarrow$ On axis, this differential contribution contribute to dB

$$dB(z) = \frac{\mu_0}{2} \frac{r^2\, dI(r)}{(r^2+z^2)^{3/2}} = \frac{\mu_0}{2} \sigma \omega \frac{r^3\, dr}{(r^2+z^2)^{3/2}}$$

$$\Rightarrow B(z) = \frac{\mu_0 \sigma \omega}{2} \int_0^R \frac{r^3\, dr}{(r^2+z^2)^{3/2}}$$

$$\boxed{B(z) = \frac{\mu_0 \sigma \omega}{2} \left[-2z + \frac{R^2 + 2z^2}{\sqrt{R^2+z^2}} \right]}$$

Look at limit $z \gg R$

$$B(z) = \frac{\mu_0 \sigma \omega}{2} z \left[-2 + \frac{2 + R^2/z^2}{\left(1+\frac{R^2}{z^2}\right)^{1/2}} \right]$$

$$\approx \frac{\mu_0}{2} \frac{\sigma \omega}{2} z \left[-2 + \left(2 + \frac{R^2}{z^2}\right)\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right) \right]$$

$$\boxed{B(z) \approx \frac{\mu_0}{8} \frac{\sigma \omega R^4}{z^3}} \sim \frac{1}{z^3} \quad \underline{\text{dipolar}}$$

Divergence and Curl of $\vec{B}$

Let us begin with the Biot - Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d^3r'$$

$$\Rightarrow \vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int d^3r' \; \vec{\nabla} \cdot \left(\vec{J}(\vec{r}\,') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}} \right)$$

recall: $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$, $\vec{\nabla}$ acts on unprimed $\vec{r}$

Using product rule: $\vec{\nabla} \cdot \left(\vec{J}(\vec{r}\,') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}} \right) = -\vec{J}(\vec{r}\,') \cdot \boxed{\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}$

But we know $\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$ since $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ is the electrostatic field of a unit point charge

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

This is the mathematical statement of "there are no magnetic monopoles". The magnetic fields never converge or diverge from a point.

Aside: This law is empirical rather than fundamental. If, some day, one were to discover "magnetic charges" we would have to modify this law.

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\vec{J}(\vec{r}\,')\times\frac{\hat{\imath}}{\imath^2}\right)\,d^3r'$$

Using product rule:

$$\vec{\nabla}\times\left(\vec{J}(\vec{r}\,')\times\frac{\hat{\imath}}{\imath^2}\right) = \vec{J}(\vec{r}\,')\left(\vec{\nabla}\cdot\frac{\hat{\imath}}{\imath^2}\right) - \cancel{(\vec{J}\cdot\vec{\nabla})\frac{\hat{\imath}}{\imath^2}}$$

See text

Now this part is tricky:

$$\frac{\hat{\imath}}{\imath^2} = 4\pi\epsilon_0 \times \text{electric field of a unit charge @ } \vec{r}\,'\text{, observed at } \vec{r}$$

$$\Rightarrow \vec{\nabla}\cdot\frac{\hat{\imath}}{\imath^2} = 4\pi\epsilon_0\frac{\rho_{\text{of point charge @ }\vec{r}\,'}(\vec{r})}{\epsilon_0}$$

$$\rho_{\text{point at }\vec{r}\,'}(\vec{r}\,') = \underset{\substack{\uparrow \\ \text{"Dirac delta function"}}}{\delta(x-x')}\,\delta(y-y')\,\delta(z-z')$$

$$\Rightarrow \vec{\nabla}\cdot\frac{\hat{\imath}}{\imath^2} = 4\pi\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\int d^3r'\,\vec{J}(\vec{r}\,')\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$ Ampere's Law