Physics 405: Lecture 25

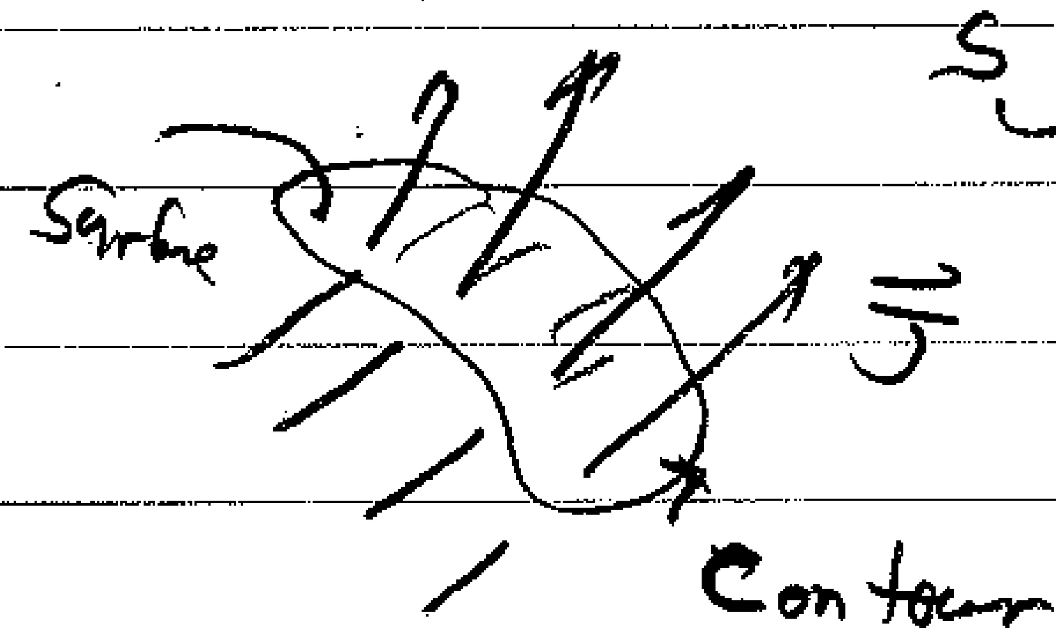
Ampere's Law

The fundamental equations of magneto-statics:

$$\vec{\nabla}\cdot\vec{B} = 0 \,, \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

- In integral form $\Rightarrow$ $\oint_S \vec{B}\cdot d\vec{a} = 0$ flux of $\vec{B}$ through closed surface is zero

- $$\int (\vec{\nabla}\times\vec{B})\cdot da = \mu_0 \underbrace{\int_S \vec{J}\cdot d\vec{a}}_{\text{"}I_{enclosed}\text{"}}$$



Stokes theorem $\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \oint_C \vec{B}\cdot d\vec{l}$ (right hand rule)

$$\Rightarrow \boxed{\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}}$$

Integral form of Ampere's Law.

Like Gauss's Law in electrostatics, in symmetrical situations, its the <u>best</u> way to find $\vec{B}$.

Canonical example: Infinite wire, current I

Symmetry and $\vec{\nabla} \cdot \vec{B} = 0$

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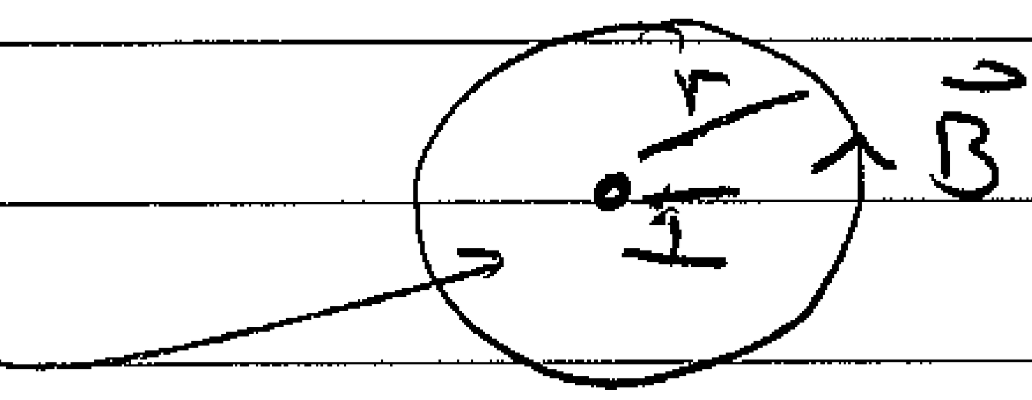


$$\Rightarrow \boxed{\vec{B} = B(r)\,\hat{\phi}}$$

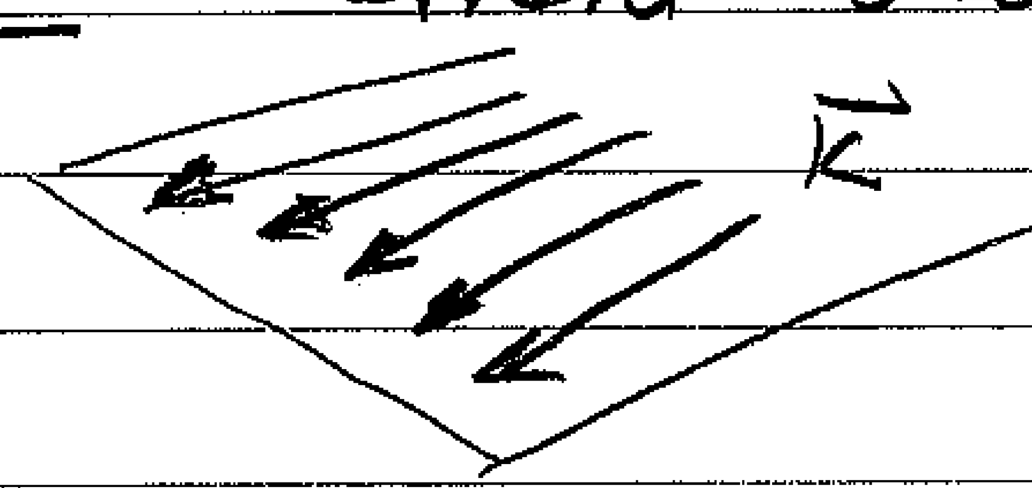
Chose contour along $\vec{B}$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \oint B(r)\, dl$$

$$= B(r) \oint l = 2\pi r\, B(r)$$

$$I_{enc} = I \Rightarrow \boxed{B(r) = \frac{\mu_0}{2\pi r} I}$$

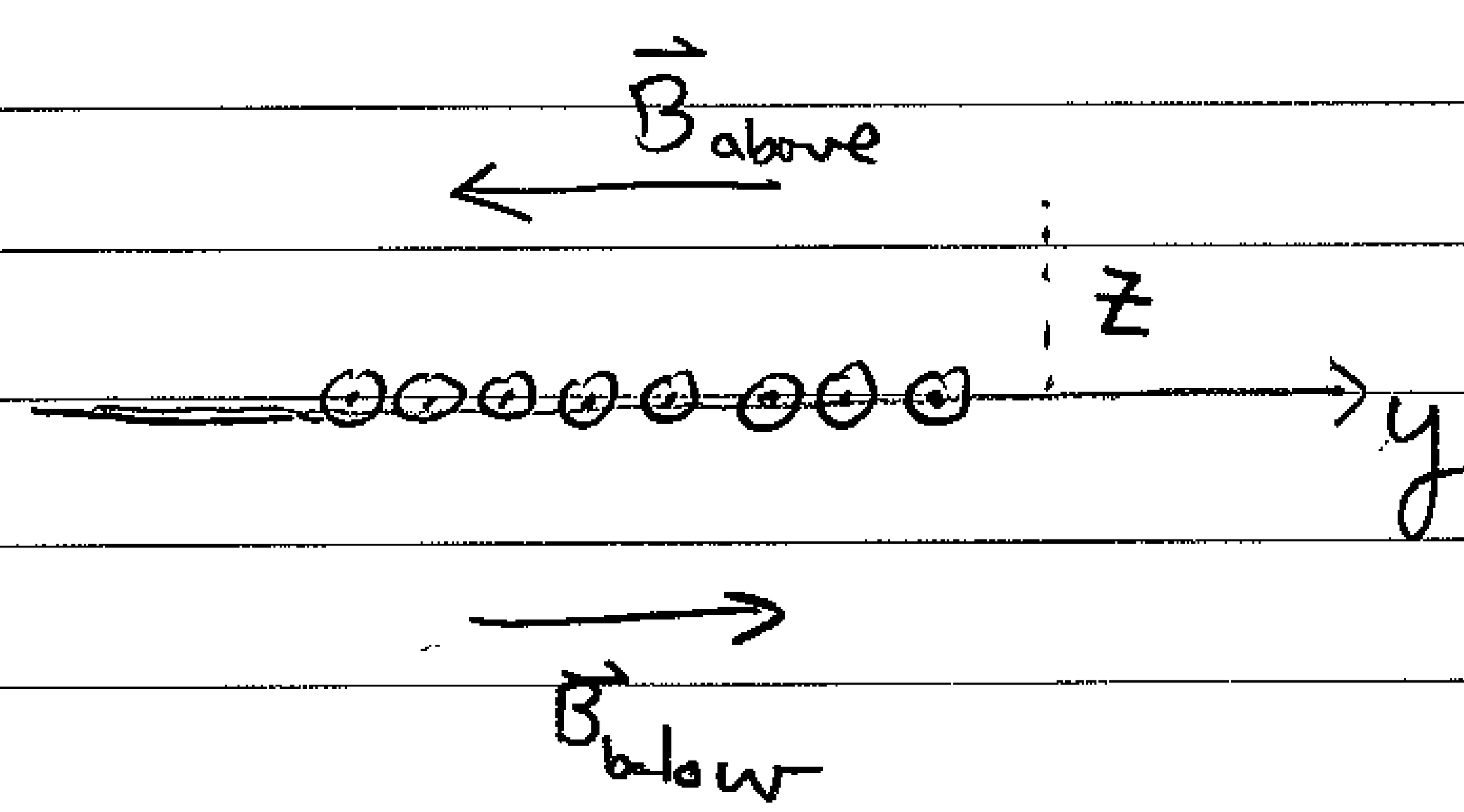
Example Current Sheet



Unform current on infinite plane

Let $\vec{K} = K\hat{x}$

Edge-on

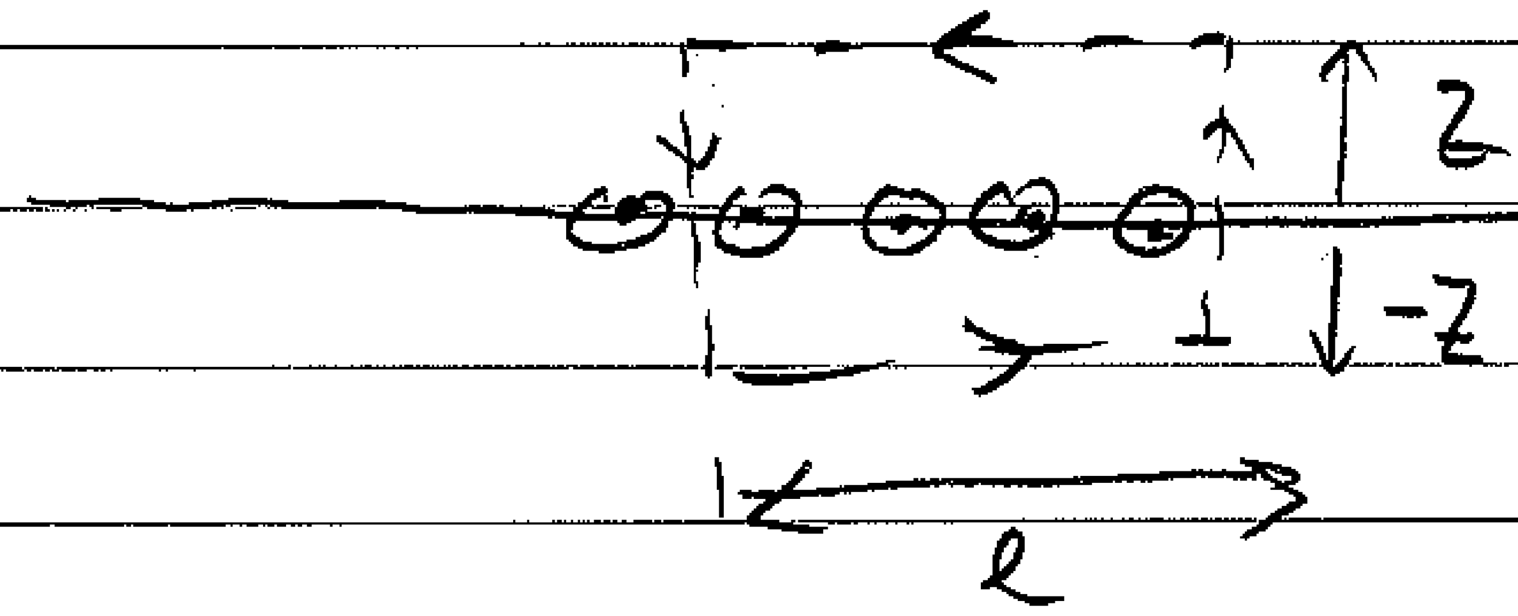


By symmetry and R.H.R $\vec{B} = B(z)\,\hat{y}$

$$B(z) = -B(-z)$$

$|\vec{B}|$ independent of x and y

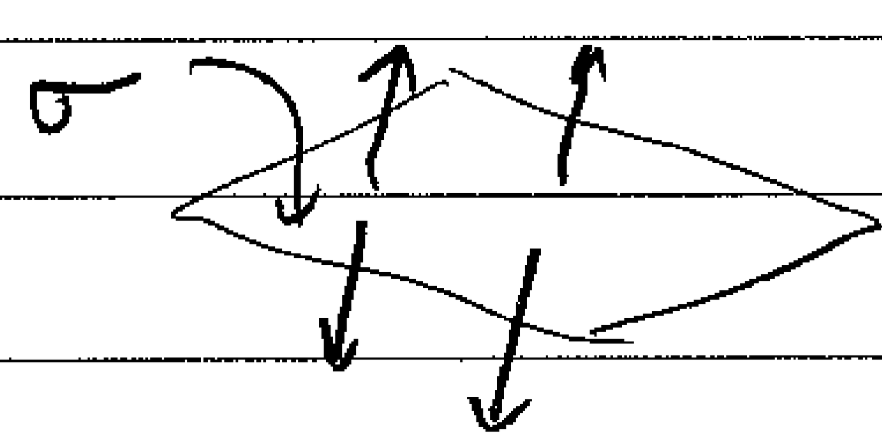
Amperian Loop



$$\oint \vec{B} \cdot d\vec{l} = 2l\,|\vec{B}(z)| = \mu_0 I_{enc}$$
$$I_{enc} = Kl$$

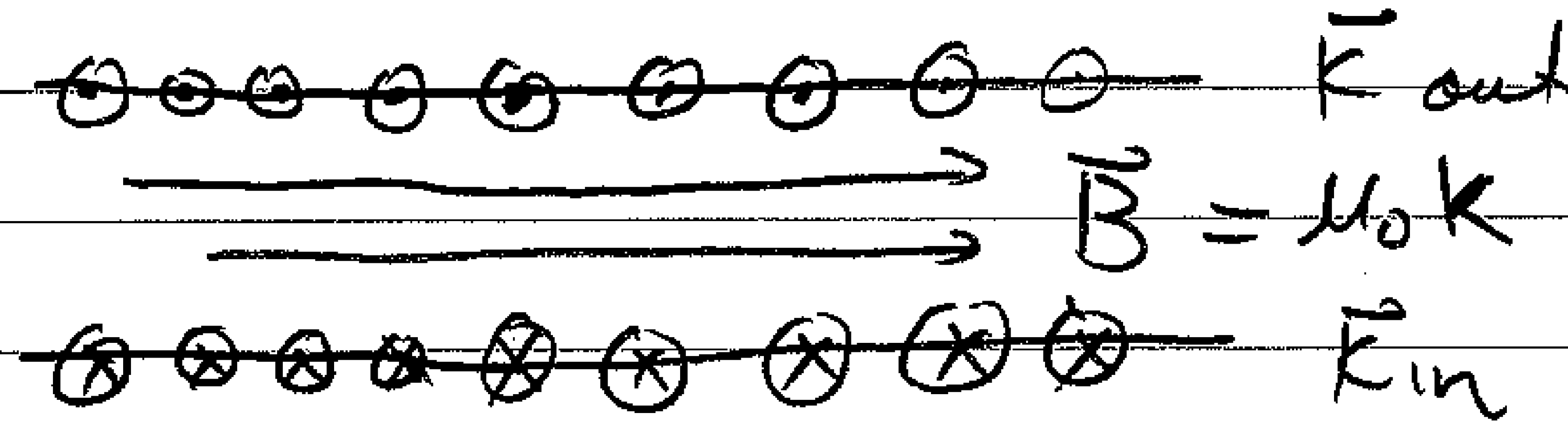
$$\Longrightarrow |\vec{B}(z)| = \frac{\mu_0 K}{2} \text{ independent of } z$$

Analogous: $\vec{E}$-field of plane of charge:

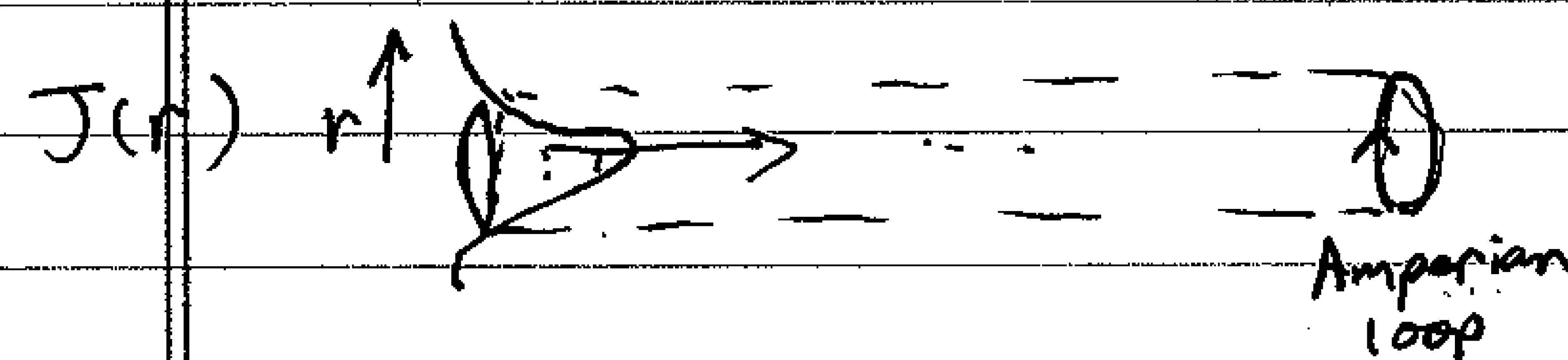


$$|\vec{E}(z)| = \frac{1}{2\epsilon_0}\sigma$$

Two planes of current (equal opposite)



Example: Cylindrical current "beam"



Azimuthal symmetry: Current density: $\vec{J}(r)$

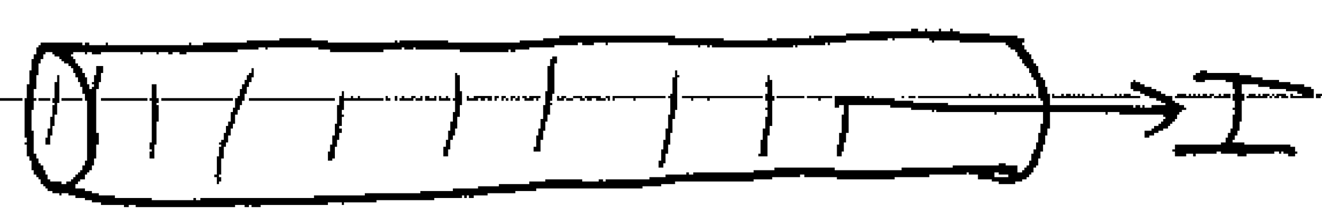
Symmetry $\Rightarrow$ $\vec{B} = B(r)\,\hat{\phi}$

$$\oint \vec{B}\cdot d\vec{l} = (2\pi r)\,B(r) = \mu_0 I_{enc}(r)$$

$$\Rightarrow B(r) = \frac{\mu_0 I_{enc}(r)}{2\pi r} = \frac{\mu_0 \int_0^r J(r')\,2\pi r'\,dr'}{2\pi r}$$

Examples:

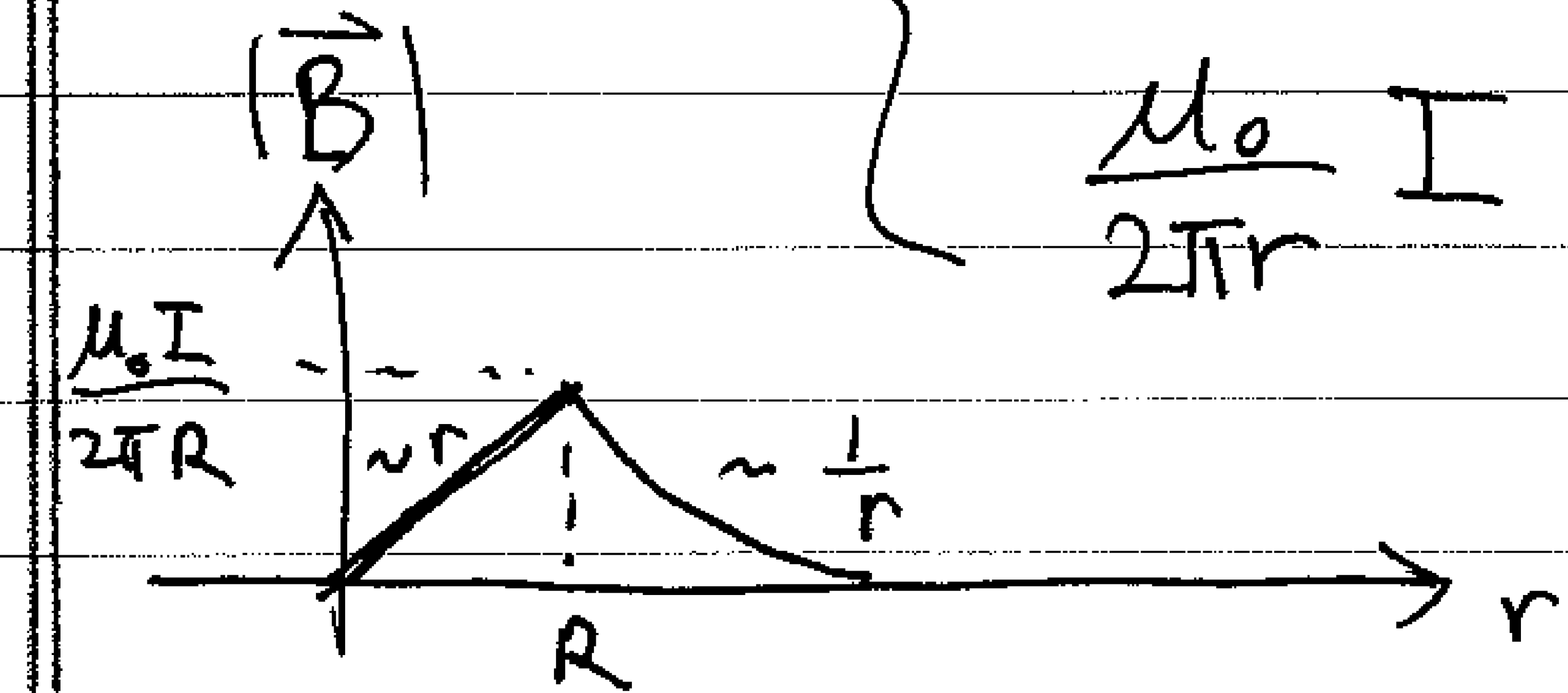
- Uniform current in a pipe, radius R



$$\Rightarrow J = \begin{cases} \dfrac{I}{\pi R^2} & r < R \\ 0 & r > R \end{cases}$$

$$\Rightarrow I_{enc}(r) = \begin{cases} \pi r^2 J = \left(\dfrac{r}{R}\right)^2 I & r < R \\ = I & r > R \end{cases}$$

$$\therefore\ B(r) = \begin{cases} \dfrac{\mu_0 r}{2\pi R^2}\, I & r < R \\[2ex] \dfrac{\mu_0}{2\pi r}\, I & r > R \end{cases}$$



Gaussian beam: $J(r) = \frac{I}{2\pi(\Delta r)^2} e^{-r^2/2(\Delta r)^2}$

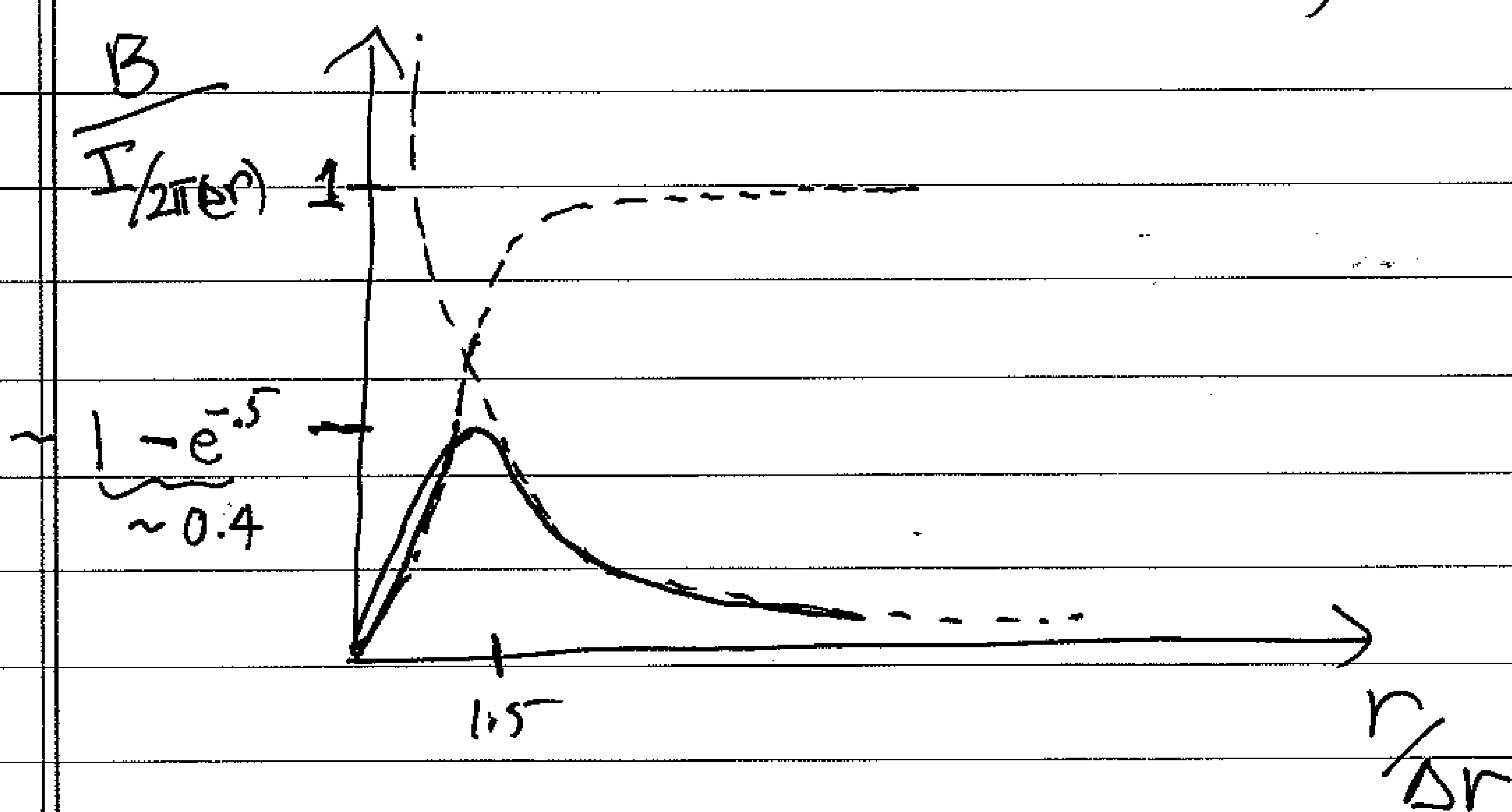
$$\int_0^\infty J(r')\, 2\pi r' dr' = I$$

$$\Rightarrow I_{enc}(r) = \int_0^r J(r')\, 2\pi r' dr' =$$

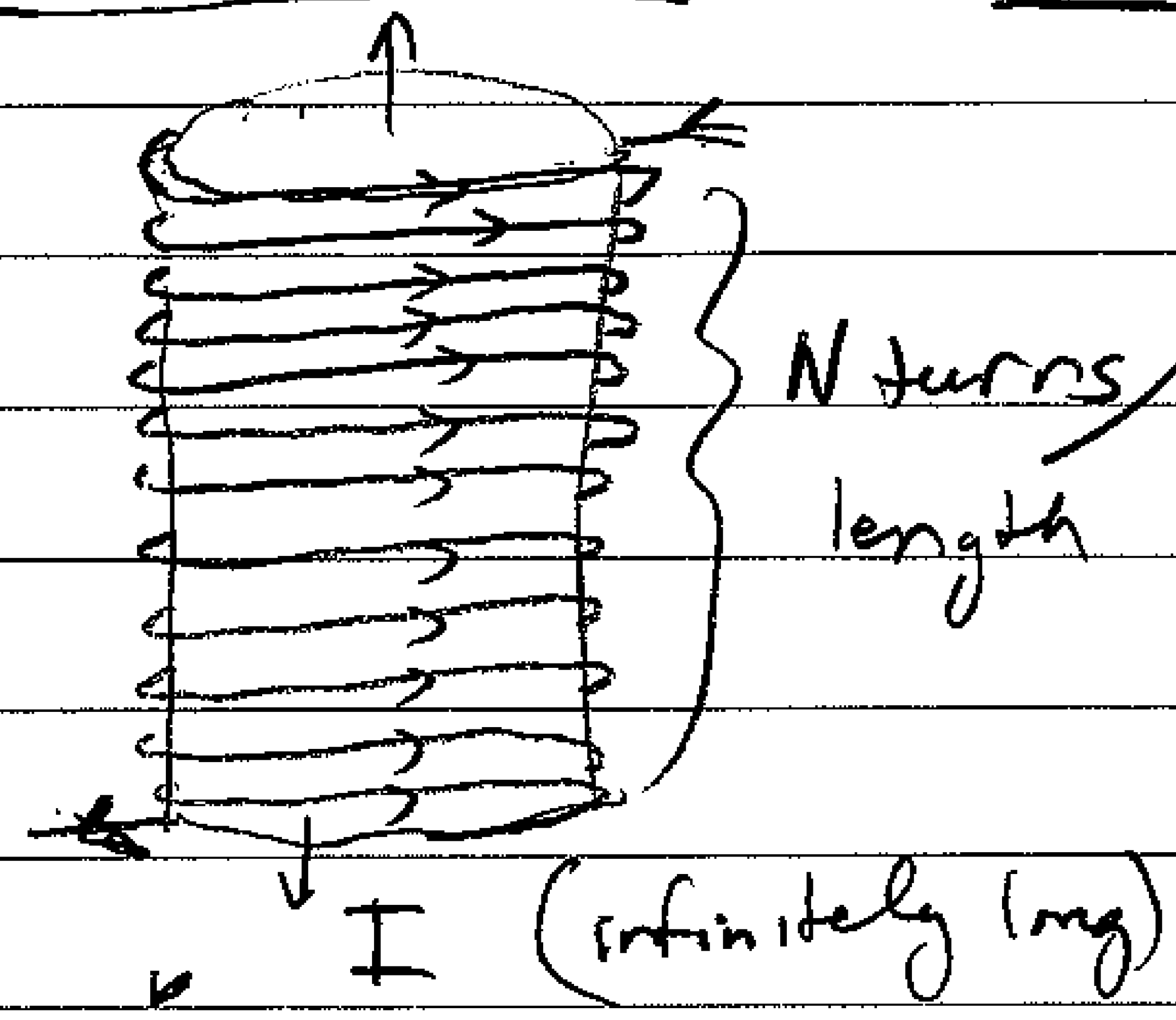
$$= \frac{I}{(\Delta r)^2} \underbrace{\int_0^r r' e^{-r'^2/2(\Delta r)^2} dr'}_{(\Delta r)^2\left(1 - e^{-\frac{r^2}{2(\Delta r)^2}}\right)}$$

$$\Rightarrow I_{enc}(r) = I\left(1 - e^{-r^2/2(\Delta r)^2}\right)$$

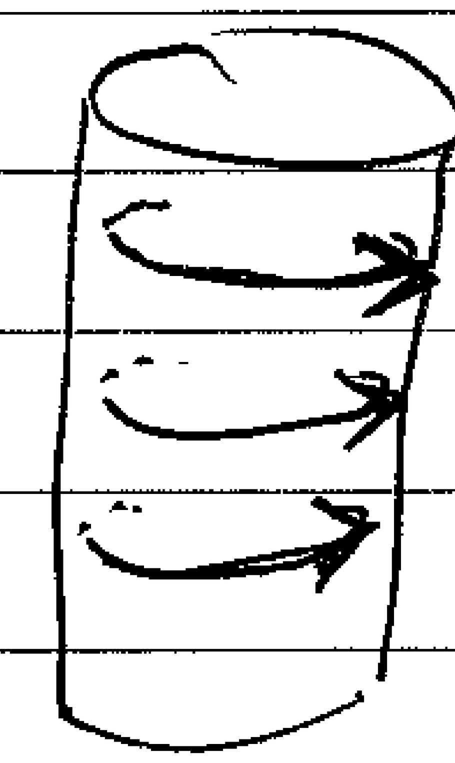
$$\therefore\ B(r) = \frac{I}{2\pi r}\left(1 - e^{\left(\frac{-r^2}{2(\Delta r)^2}\right)}\right)$$



Important Example: The Solenoid



$\Rightarrow$



Direction of $\vec{B}$?

Cylindrical components: $\vec{B} = B_r\,\hat{r} + B_\phi\,\hat{\phi} + B_z\,\hat{z}$

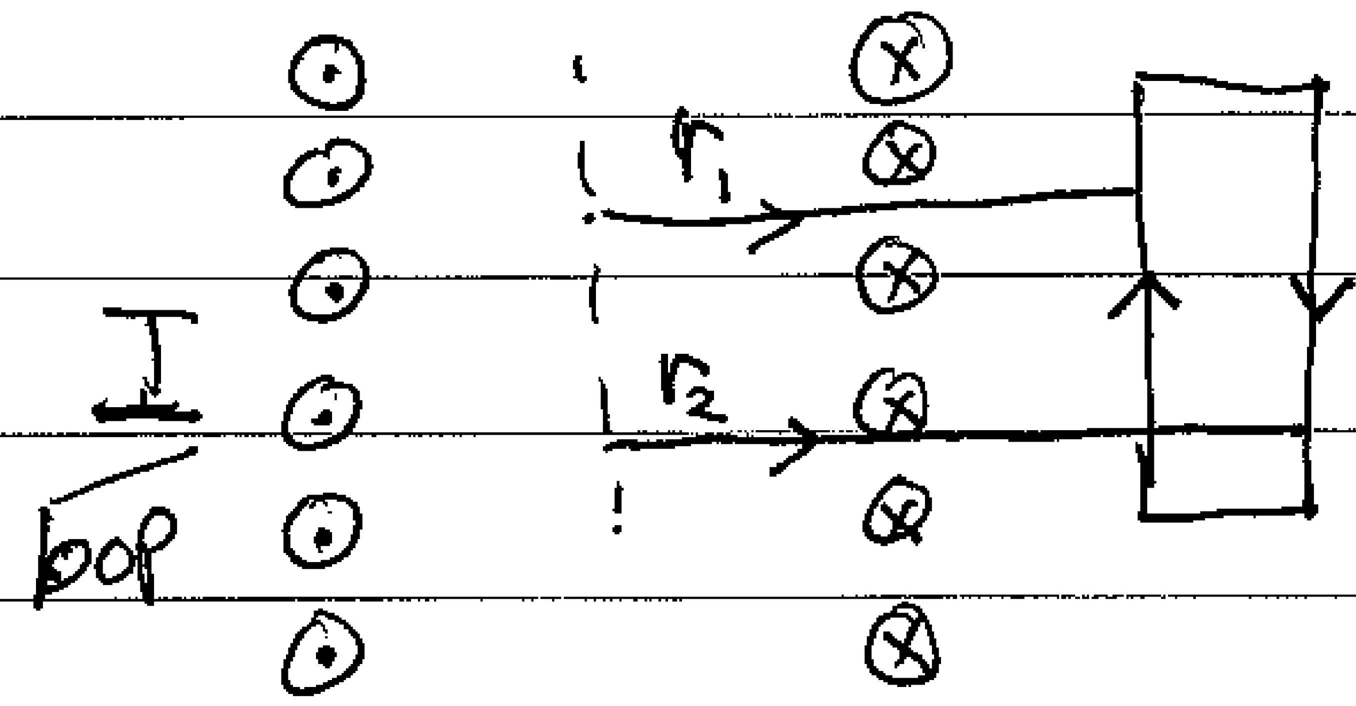
- By symmetry $B_\phi = 0$, since current in $\hat{\phi}$ direction

$|\vec{B}|$ is independent of z (infinitely long)
independent of ϕ (azimuthally symmetric)

Since $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow B$ has no $\hat{r}$ component

$$\Rightarrow \boxed{\vec{B} = B_z(r)\,\hat{z}}$$

• Outside Solenoid:

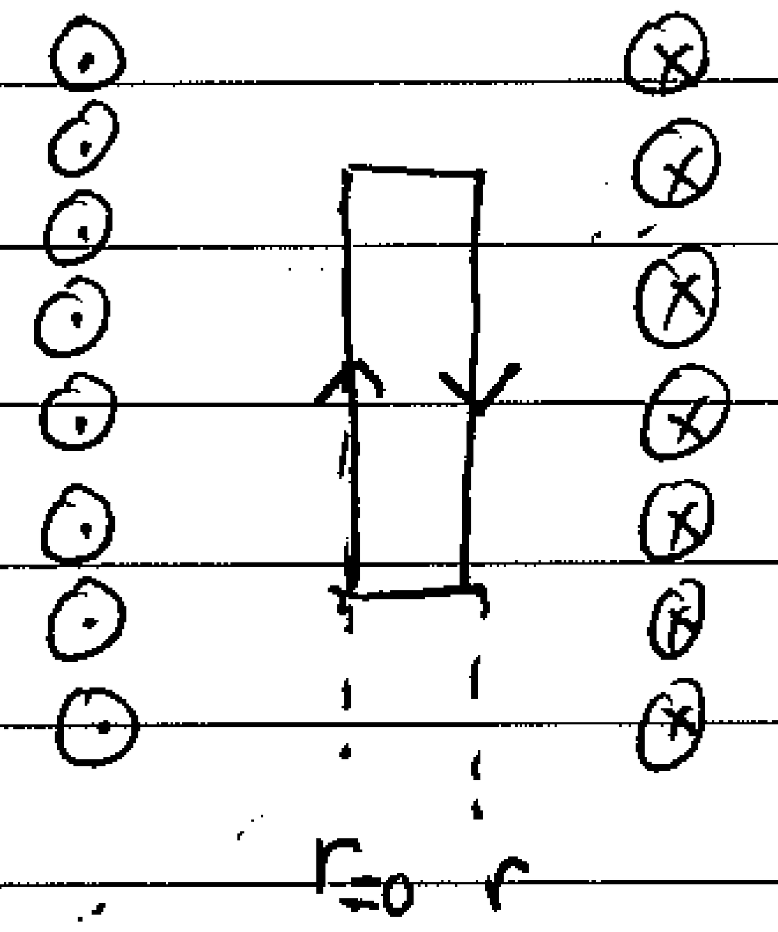


$$\oint \vec{B}_{out}\cdot d\vec{l} = B_{out}(r_1) - B_{out}(r_2) = 0$$

$$\Rightarrow B_{out}(r_1) = B_{out}(r_2)$$

But: $B_{out} \to 0$ as $r \to \infty$

$$\therefore \boxed{B_{out} = 0}$$

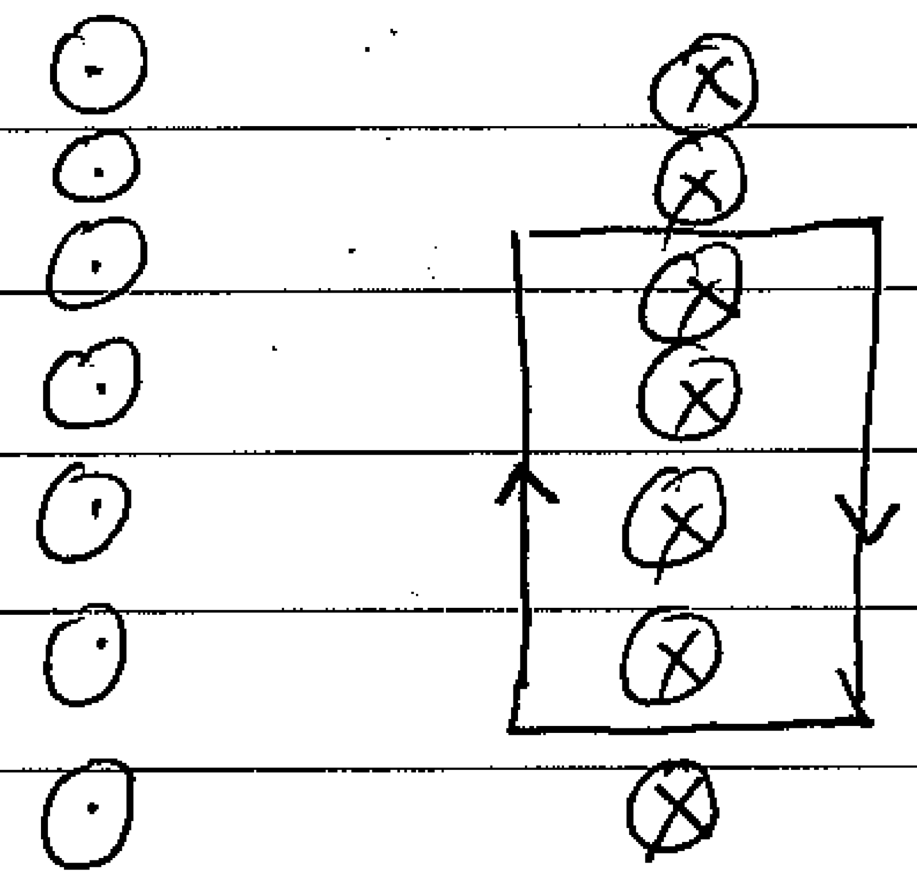


Inside:

$$\oint \vec{B}_{in} \cdot d\vec{\ell} = B_{in}(0) - B_{in}(r) = 0$$

$$\Rightarrow B_{in}(r) = B_{in}(0)$$

Now consider a loop which straddles inside + outside



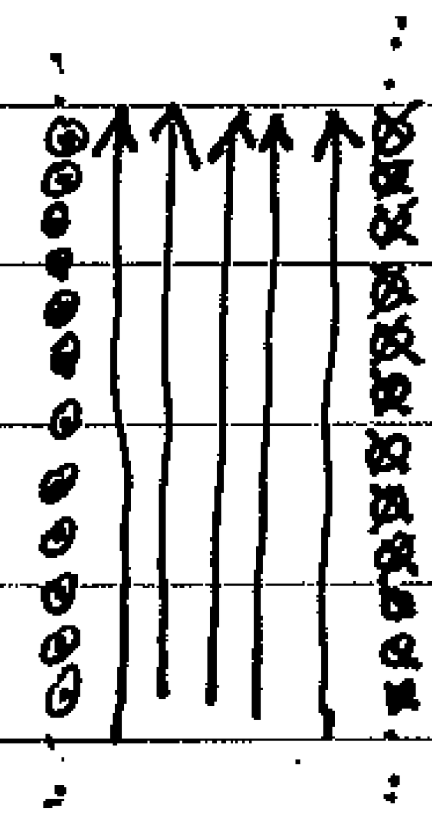
$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 N I \ell$$

$$\Rightarrow \ell(B_{in} - \underbrace{B_{out}}_{0}) = \mu_0 N I \ell$$

$$\Rightarrow \boxed{\vec{B}(r) = \begin{cases} \mu_0 N I \hat{z} & \text{inside} \\ 0 & \text{outside} \end{cases}}$$

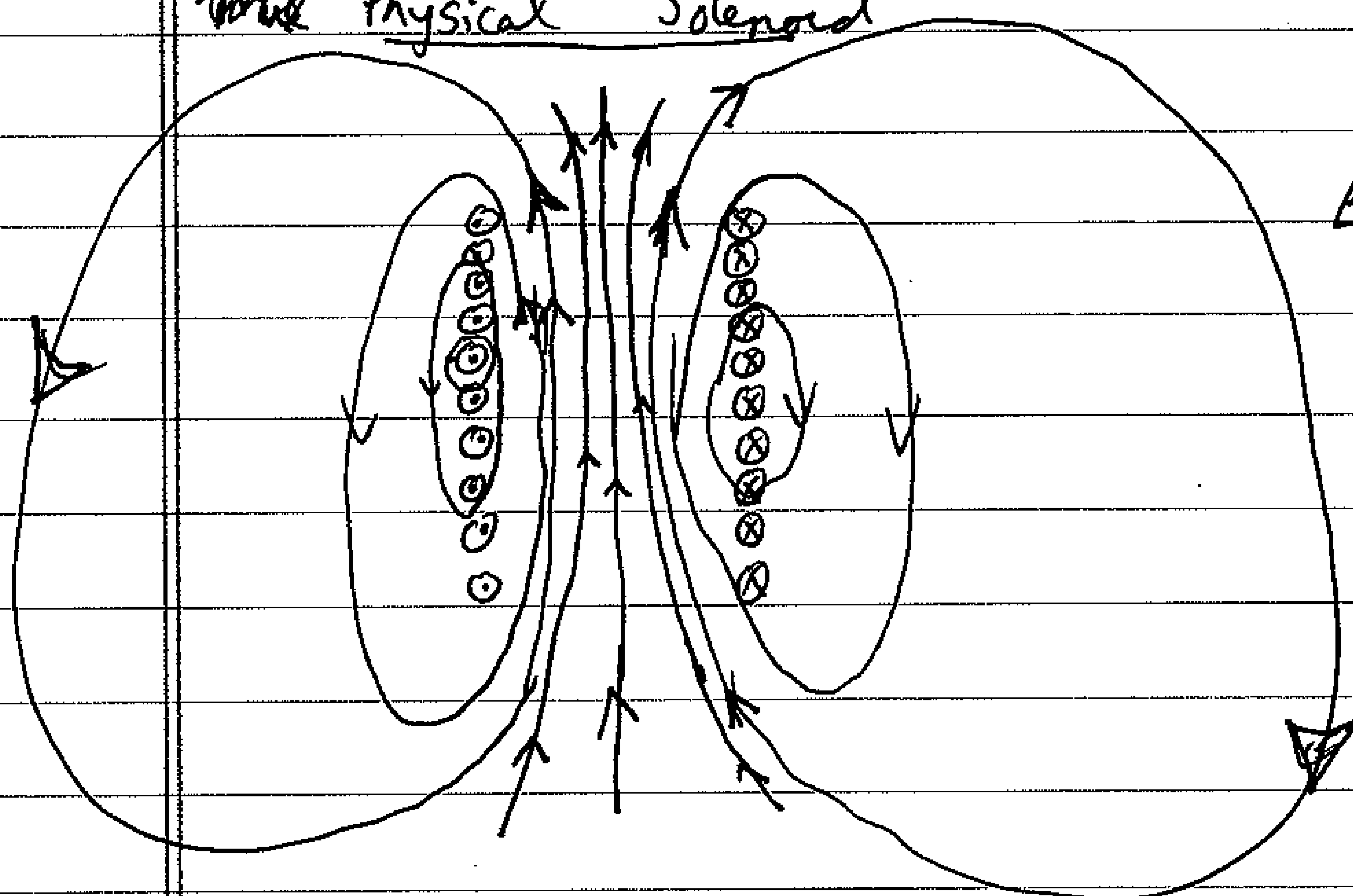
N = # of turns/length

Ideal Solenoid



$\vec{B}$ perfectly uniform inside,
zero outside

The Physical Solenoid



Fringing fields

$|\vec{B}|_{outside} << |\vec{B}|_{inside}$

When diameter << length

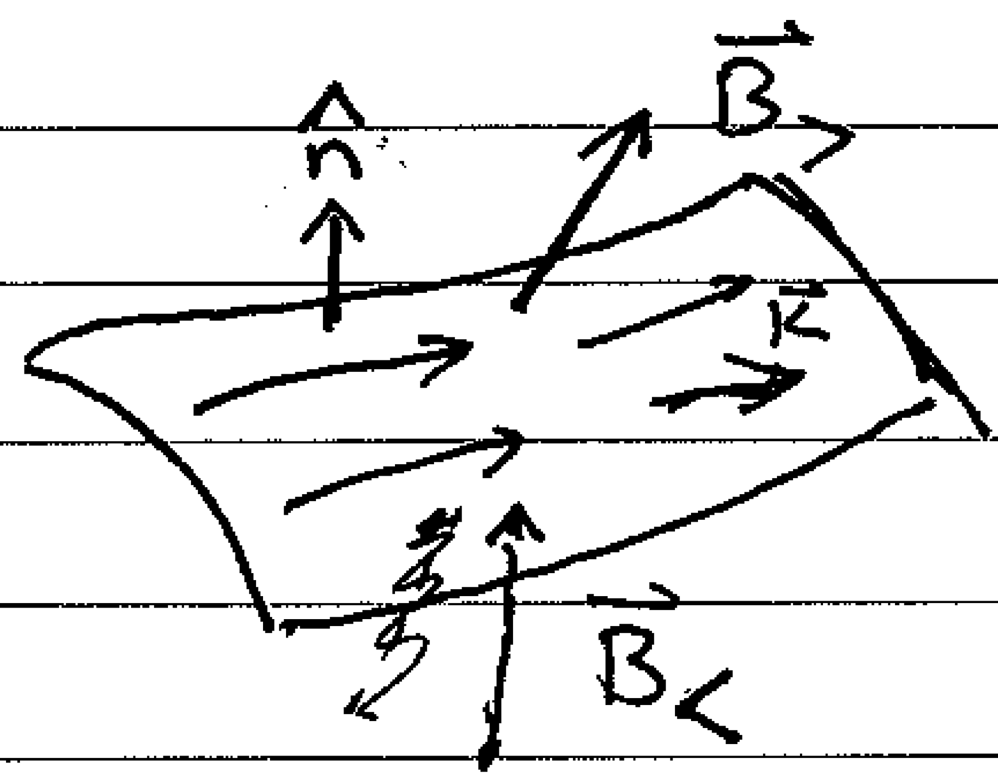
Parallel plate capacitor as Solenoid

Electro-statics Magneto-statics

Boundary Conditions on $\vec{B}$ (Magneto-Statics)

Fundamental field eqns: $\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \end{cases}$

Consider a surface dividing two regions:

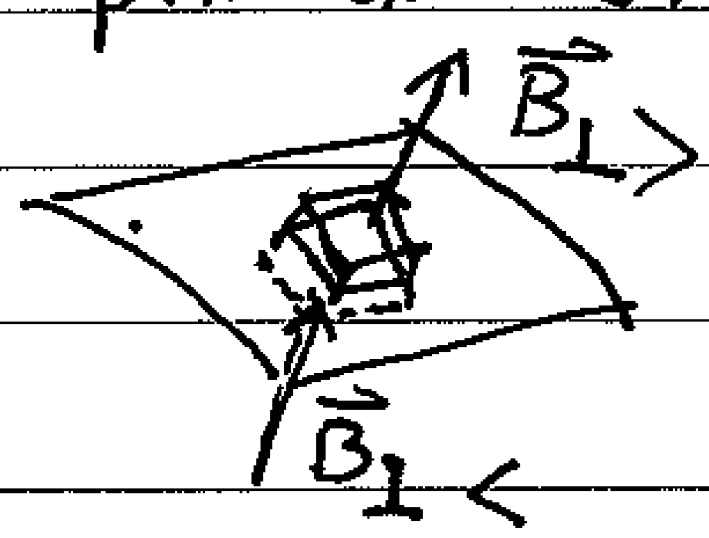


We consider the possibility of current flowing on the surface: $\vec{K}$ = surface current density

To determine the b.c. on the normal component

$$B_\perp \equiv \hat{n}\cdot\vec{B}$$

Take a "pill box" straddling the surface



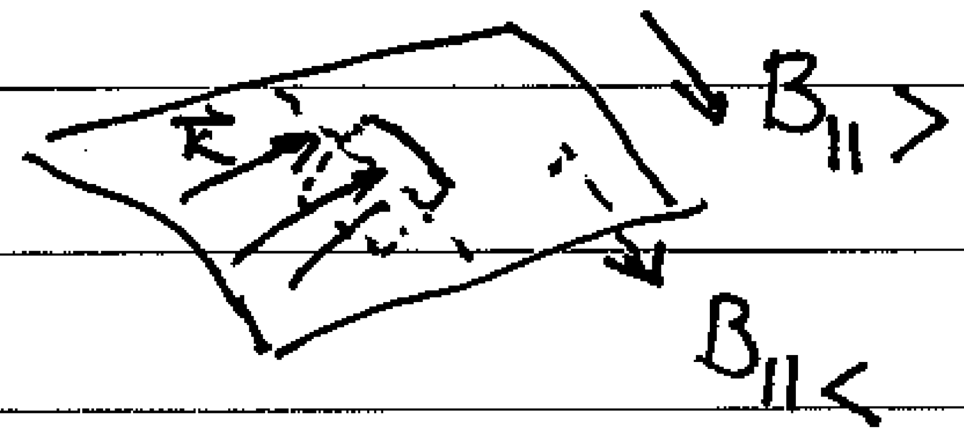
$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \oint \vec{B}\cdot d\vec{a} = 0$$

$$A(B_{\perp >} - B_{\perp <}) = 0$$

$$\Rightarrow \boxed{B_{\perp >} = B_{\perp <}}$$

To determine the b.c. on the tangential component

Take an Amperian loop straddling the surface



$$\oint \vec{B}\cdot d\vec{l} = l(B_{\parallel >} - B_{\parallel <})$$

$$= \mu_0 I_{enc} = \mu_0 K l$$

$$\Rightarrow \boxed{B_{\parallel >} - B_{\parallel <} = \mu_0 K}$$