# Physics 405: Lecture 26

## The Vector Potential

In electrostatics the equation

$$\vec{\nabla}\times\vec{E}=0 \Rightarrow \boxed{\vec{E}=-\vec{\nabla}V}$$

~~[illegible]~~ Where $V$ is the "electrostatic potential"

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho(\vec{r})}{\epsilon_0} \Rightarrow \boxed{\nabla^2 V=-\frac{\rho(\vec{r})}{\epsilon_0}}$$ Poisson's Equation

Solutions to electrostatics is thus the solution to Poisson's Eqn. For localized $\rho(\vec{r})$ with ground @ infinity

$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

For a ~~[illegible]~~ given $\rho(\vec{r})$ it is generally simpler to find $V(\vec{r})$ than $\vec{E}(\vec{r})$ (no vectors to deal with). Moreover, often we are interested in the field away from the charges where $\rho(\vec{r})=0$

$$\Rightarrow \nabla^2 V=0 \quad \text{(Laplace's Eqn).}$$

Then we have a variety of methods to find the solution using boundary values.

In magneto-statics

$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ . Thus, in general $\vec{B} \neq -\vec{\nabla} V_{mag}$

Magnetic forces are not "conservation".
Magnetic fields do no work.

To see this, look at the Lorentz Force:

$$\vec{F} = q\vec{v}\times\vec{B} = m\vec{a}$$

(point particle, mass m, charge q)

The rate at which work is done on a particle

$$\frac{dW}{dt} = \vec{v}\cdot\vec{F} = \vec{v}\cdot(q\vec{v}\times\vec{B}) = 0$$

Said another way, consider the kinetic energy. The rate at which the kinetic energy changes due to magnetic forces

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = m\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$= m\vec{a}\cdot\vec{v} = \vec{F}\cdot\vec{v} = 0$$

Magnetic forces can change the direction of $\vec{v}$, but not it's magnitude

$$\boxed{\vec{B}\text{-fields do not work}}$$

The magnetic field is, however, mathematically related to a "vector potential"

$$\vec{\nabla}\cdot\vec{B} = 0 \iff \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$$

It is easy to check $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$, but it also follows that if $\vec{\nabla}\cdot\vec{B}=0$ then $\exists\, \vec{A}(\vec{r})$ s.t. $\vec{B} = \vec{\nabla}\times\vec{A}$ (Helmholtz theorem).

$$\boxed{\vec{A}(\vec{r}) \equiv \text{Vector Potential}}$$

The vector potential, unlike the scalar potential in electrostatics, is not interpretable in terms of potential energy. It is a calculational tool.

Plugging $\vec{B} = \vec{\nabla}\times\vec{A}$ into Ampère's Law

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0 \vec{J}$$

Aside: Vector identity $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$$\Rightarrow \boxed{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}}$$

Yikes!

"Gauge Invariance"

In electro-statics, the potential is defined up to a constant (the point of "ground")

There is much more freedom in defining the vector potential for a given $\vec{J}$.

Note: If $\vec{F} = \vec{\nabla} \chi(\vec{r})$, for an arbitrary scalar field $\chi(\vec{r})$

then $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (\vec{\nabla} \chi) = 0$

Thus, we can add to $\vec{A}(\vec{r})$ an arbitrary vector field $\vec{\nabla}\chi$, without changing $\vec{B}$

$$\vec{A} \Rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi \; : \; \underline{\text{Gauge Transformation}}$$

$$\underbrace{\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}'}_{\text{mathematical entity}} = \vec{B} \; : \; \text{Physical field}$$

Said another way: The vector potential $\vec{A}$ is determined by its curl and its divergence. The curl of $\vec{A}$ defines the physical magnetic field, but its divergence is arbitrary

Choice of "gauge": In Magnetostatics, we typically choose: $\boxed{\vec{\nabla} \cdot \vec{A} = 0}$

With this choice of gauge.

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

Each Cartesian component of $\vec{A}$ satisfies a Poisson Eqn, with source determined by the Cartesian component of $\vec{J}$

$$\Rightarrow \boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}}$$

For a steady surface or line current:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int da \, \frac{\vec{K}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \frac{\mu_0 I}{4\pi} \int d\vec{l}\,' \frac{1}{|\vec{r} - \vec{r}\,'|}$$

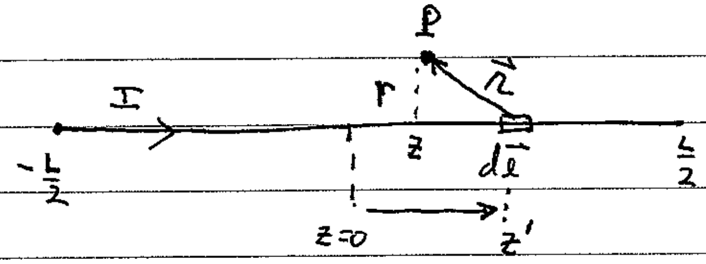
In contrast to the electrostatic scalar potential, the magnetostatic vector potential is not as useful a calculational tool because the argument of the integral is still a vector. In electrodynamics as we will see in 406, the vector potential is central. More often, in practical calculations of $\vec{B}$, we want to find the field in a region where $\vec{J} = 0$

$\Longrightarrow$ if @ point of observation $\vec{J} = 0$

$$\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla} U_m \Leftarrow \text{magnetic "pseudo potential"}$$

One must be careful about connecting different "simply connected" regions, but very useful practically.

Example: Vector Potential of a Line Charge




$$\vec{A}(r,z) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}}{\mathscr{r}} = \frac{\mu_0 I}{4\pi}\hat{z}\int_{-L/2}^{L/2} \frac{dz'}{\sqrt{r^2+(z-z')^2}}$$

$$\Rightarrow \boxed{\vec{A}(r,z) = \frac{\mu_0 I}{4\pi}\hat{z}\,\ln\left[\frac{L+2z+\sqrt{L^2-4Lz+4(r^2+z^2)}}{-L+2z+\sqrt{L^2+4Lz+4(r^2+z^2)}}\right]}$$

Not too pretty

Using Curl in Cylindrical coords

$$\vec{B}(r,z) = \vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial r}\hat{\phi} = \underline{\text{A mess!}}$$

limit of an infinite wire : Set $z=0$,

$$\vec{A}(r,z=0) = \frac{\mu_0 I}{4\pi}\hat{z}\,\ln\left[\frac{L+\sqrt{L^2+4r^2}}{-L+\sqrt{L^2+4r^2}}\right]$$

$$\lim_{L\to\infty} \vec{A}(r,z=0) = \frac{\mu_0 I}{4\pi}\hat{z}\,\ln\left[\frac{2}{-1+(1+\frac{4r^2}{L^2})^{1/2}}\right] \approx \frac{\mu_0 I}{4\pi}\hat{z}\,\ln\left[\frac{L^2}{r^2}\right]$$

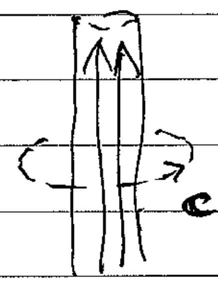
$$\lim_{L\to\infty} \boxed{\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0 I}{2\pi r}\hat{z}}$$ As expected

## Vector Potential of a Solenoid

Consider infinite solenoid with N turns/length.
Instead of finding $\vec{A}$ from current, we can find it from $\vec{B}$ by integration:

Note: $$\int_S \vec{B}\cdot d\vec{a} = \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_C \vec{A}\cdot d\vec{l}$$

The contour integral of $\vec{A}$ around a closed loop is the <u>flux</u> of $\vec{B}$ through the loop




$\Rightarrow$ Use reasoning of Ampère's Law

$$\vec{\nabla}\times\vec{A} = \vec{B} \quad \text{with } \vec{\nabla}\cdot\vec{A} = 0$$

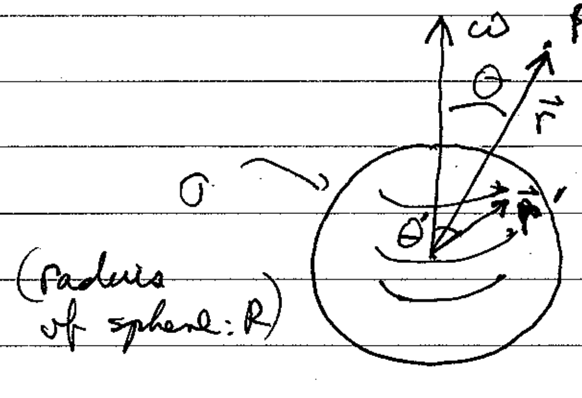
$\Rightarrow$ $\vec{B}$ uniform in $z$-direction

$\vec{A}$ in $\hat{\phi}$-direction, varying w/ $r$

$$\Rightarrow 2\pi r\, A(r) = \begin{cases} B(\pi r^2) & r < R \\ B(\pi R^2) & r > R \end{cases}$$

$$\Rightarrow \boxed{\vec{A}(r) = Br\hat{\phi} = \begin{cases} \hat{\phi}\,\dfrac{\mu_0 N I}{2}\, r & r < R \\ \hat{\phi}\,\dfrac{\mu_0 N I}{2}\,\dfrac{R^2}{r} & r > R \end{cases}}$$

Example: Spherical shell of charge Rotating at angular velocity $\omega$




Surface charge density

$$\vec{K}(\theta') = \sigma \vec{v}(\theta')$$

$$\vec{v}(\theta') = \vec{\omega} \times \vec{R}'$$

$$= \omega R \sin\theta' \, \hat{\phi}$$

Note: Since $|\vec{K}|$ varies with $\theta'$ and points in $\hat{\phi}$ direction, so simple symmetry to use Ampère's law

Find vector ~~field~~ potential @ point of observation P

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}')\, da'}{|\vec{r} - \vec{r}'|}$$

$$da' = R^2 \sin\theta' d\theta' d\phi' \quad \cancel{\neq \underbrace{2\pi}_{\text{azimuthal symmetry}} R^2 \sin\theta' d\theta'}$$

Note: $\hat{\phi} = -\sin\phi' \hat{x} + \cos\phi' \hat{y}$ (depends on $\phi'$)

$$\Rightarrow \boxed{\vec{A}(\vec{r}) = \frac{\mu_0 \sigma R^3 \omega}{4\pi} \int \frac{\sin^2\theta' d\theta' (-\sin\phi' \hat{x} + \cos\phi' \hat{y}) d\phi'}{\sqrt{r^2 + R^2 - 2(rR)\, \hat{r}\cdot\hat{r}'}}}$$

Yick!

Transforming coordinates, so that P along symmetry axis, Griffiths was able so solve this integral (see Text, example Ex: 11)

Solution:

$$\boxed{\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\phi} & r < R \\[2ex] \dfrac{\mu_0 R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\hat{\phi} & r > R \end{cases}}$$

Curl in Spherical

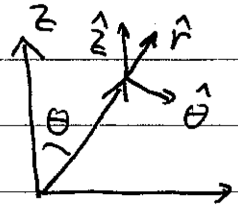
$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

Here $\vec{A} = A(r,\theta)\hat{\phi} \Rightarrow \boxed{\vec{B} = \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)}$

$\Rightarrow r \le R$

Inside

$$\vec{B}(r,\theta) = \frac{2\mu_0 R\omega\sigma}{3}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

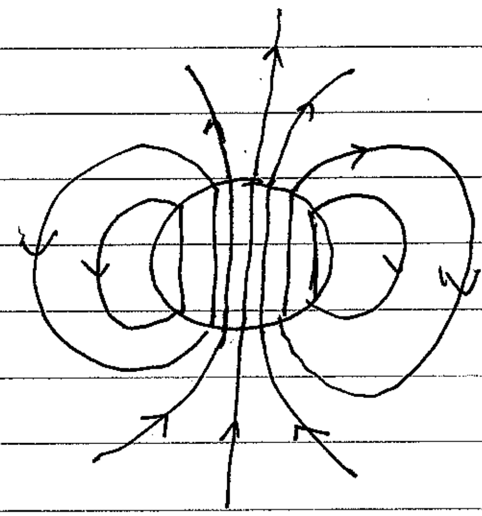



$$\Rightarrow_{r\le R} \vec{B}(r,\theta) = \frac{2\mu_0 R\omega\sigma}{3}\hat{z} = \frac{2\mu_0 R\sigma}{3}\vec{\omega} \quad \text{uniform}$$

Outside
$r \geq R$

$$\vec{B}(r,\theta) = \frac{\mu_0 R^4 \omega \sigma}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \text{Dipole Field} \sim \frac{1}{r^3}$$



Spining sphere of charge

$\Longrightarrow$ Perfect magnetic dipole