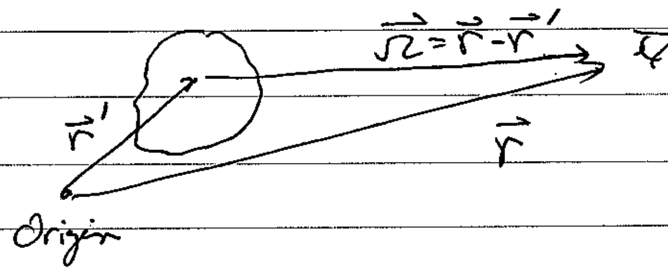
Physics 405 : Lecture 27

Multipole Expansions in Magnetostatics

Recall, in electrostatics, for a localized charge distribution (ground @ ∞)



When $\vec{r} >> \vec{r}\,'$ $\forall \vec{r}\,'$

Then the electrostatic potential can be approximately

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|} \approx \frac{1}{4\pi\epsilon_0}\left(\frac{q_{Net}}{r} + \vec{p}\cdot\frac{\hat{r}}{r^2} + Q_{ij}\,\frac{\hat{r}_i\hat{r}_j}{r^3} + \dots \right)$$

The potential decomposed into characteristic potentials that fall off as $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \dots$

determined by characteristics of the charge distribution, the multipole moments:

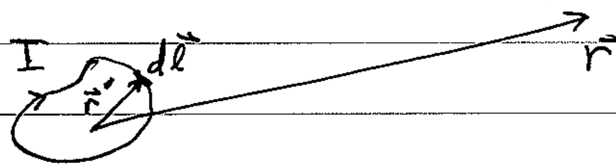
$q_{Net} = \int d^3r'\,\rho(\vec{r}\,')$: Monopole Moment

$\vec{p} = \int d^3r'\,\vec{r}\,'\rho(\vec{r}\,')$: (Electric) Dipole Moment Vector

$Q_{ij} = \int d^3r'\,\frac{1}{2}\left(3r_i'r_j' - |\vec{r}\,'|^2\delta_{ij}\right)\rho(\vec{r}\,')$ | Electric Quadrupole Tensor

We seek a similar expansion for the magnetic field. Start with the formal solution to the vector potential (in the gauge $\vec{\nabla}\cdot\vec{A} = 0$) for a confined current loop

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{d^3\vec{r}\,'\ \vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}}{|\vec{r}-\vec{r}\,'|}$$



In the limit $|\vec{r}| >> |\vec{r}\,'|$ $\qquad \frac{1}{|\vec{r}-\vec{r}\,'|} \approx \frac{1}{r} + \frac{\hat{r}\cdot\vec{r}\,'}{r^2}$

to first order in $\frac{r'}{r}$ (see Lecture).

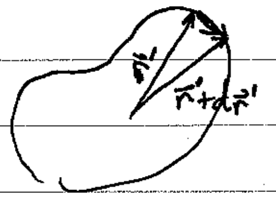
$$\Rightarrow \vec{A}(\vec{r}) \cong \underbrace{\frac{1}{r}\left(\frac{\mu_0 I}{4\pi}\oint d\vec{l}\right)}_{\text{Monopole contribution}} + \underbrace{\frac{1}{r^2}\left(\frac{\mu_0 I}{4\pi}\oint \hat{r}\cdot\vec{r}\,'\, d\vec{l}\right)}_{\text{dipole contribution}}$$

Aside

- $\oint d\vec{l} = 0$ (vectors close on themselves in a closed loop)

$\Longrightarrow$ Monopole term vanishes

- To consider the dipole term, first note



along the loop $d\vec{r}\,' = d\vec{l}$

Aside:

Consider a differential of $(\hat{r}\cdot\vec{r}\,')\vec{r}\,'$ along loop (with observation vector $\vec{r}$ fixed)

$$d[(\hat{r}\cdot\vec{r}\,')\vec{r}\,'] = (\hat{r}\cdot\vec{r}\,')d\vec{r}\,' + (\hat{r}\cdot d\vec{r}\,')\vec{r}\,'$$

$$\oint d[(\hat{r}\cdot\vec{r}\,')\vec{r}\,'] = 0$$ (perfect differential around the closed loop)

$$\Rightarrow \oint (\hat{r}\cdot\vec{r}\,')d\vec{r}\,' = -\oint (\hat{r}\cdot d\vec{r}\,')\vec{r}\,'$$

Now consider $\hat{r}\times(\vec{r}\,'\times d\vec{r}\,') = \vec{r}\,'(\hat{r}\cdot d\vec{r}\,') - (\hat{r}\cdot\vec{r}\,')d\vec{r}\,'$

$$\Rightarrow \oint \hat{r}\times(\vec{r}\,'\times d\vec{r}\,') = -2\oint(\hat{r}\cdot\vec{r}\,')d\vec{r}\,'$$

$$\Rightarrow \vec{A}_{dipole} = \frac{1}{r^2}\left(\frac{\mu_0 I}{4\pi}\right)\left(-\frac{1}{2}\oint \hat{r}\times(\vec{r}\,'\times \underset{=d\vec{r}\,'}{d\vec{l}})\right)$$

$$\therefore \boxed{\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\,\vec{m}\times\frac{\hat{r}}{r^2}}$$

where $\vec{m} = I\oint \vec{r}\,'\times d\vec{l}$ (current loop)

$= \int \vec{r}\,'\times\vec{J}(\vec{r}\,')\,d^3r'$ (general distribution)

$\vec{m}$ = magnetic dipole moment

The magnetic field associated with the "dipole vector potential"

$$\vec{B}_{dipole} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\vec{m} \times \frac{\hat{r}}{r^2} \right)$$

$$= \frac{\mu_0}{4\pi} \left(\vec{m} \underbrace{\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right)}_{0 \text{ away from } \vec{r}=0} - (\vec{m} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2} \right)$$

$$= \frac{\mu_0}{4\pi} \left(-\vec{\nabla} \cdot \left(\frac{\vec{m} \cdot \hat{r}}{r^2} \right) \right)$$

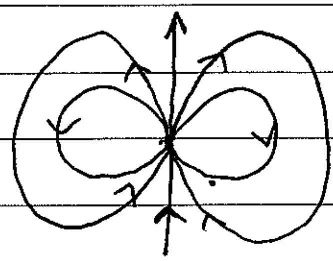
Same form as the $\vec{E}$-field of an electric dipole $\vec{E} \sim -\vec{\nabla}\left(\frac{\vec{p} \cdot \hat{r}}{r^2} \right)$

$$\Rightarrow \boxed{\vec{B}_{dipole} = \frac{\mu_0}{4\pi} \left(\frac{3(\vec{m} \cdot \hat{r})\, \hat{r} - \vec{m}}{r^3} \right)}$$

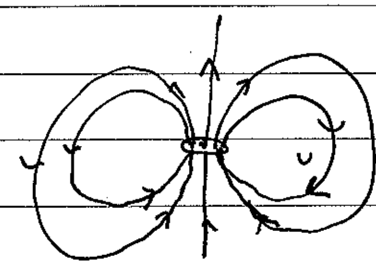
With $\vec{m}$ along z-axis, and using spherical coordinates

$$\vec{B}_{dipole} = \frac{\mu_0 m}{4\pi r^3} \left(3 \cos\theta\, \hat{r} - \hat{z} \right)$$

$$= \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right)$$

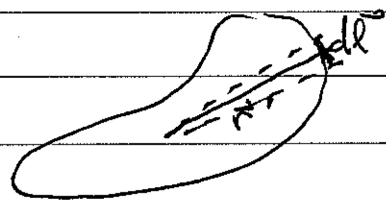


Pure dipole field



Physical magnetic dipole

Special form of $\vec{m}$ for a planar current loop



$\vec{r}' \times d\vec{l} = (\text{base}) \times (\text{height})$ of dotted triangle

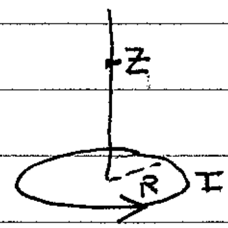
$$\Rightarrow \tfrac{1}{2}(\vec{r}' \times d\vec{l}) = da\, \underbrace{\hat{n}}_{}\ (\text{normal to loop})$$

$$\Rightarrow \frac{I}{2} \oint \vec{r}' \times d\vec{l}' = I \oint da\, \hat{n} = I\,(\text{Area})\,\hat{n}$$

$\Rightarrow$ For a planar current loop:

$$\boxed{\vec{m} = I\,(\text{Area of loop})\ \underbrace{\hat{n}}_{\text{normal by right-hand rule}}}$$

Example: Circular loop of current



The magnetic dipole moment $\vec{m} = (\pi R^2) I \hat{z}$

In Lecture 23, we found the exact $\vec{B}$-field on z-axis

$$\vec{B}(z) = \frac{\mu_0}{4\pi} \left[\frac{2\pi R^2 I}{(R^2+z^2)^{3/2}} \right] \hat{z}$$

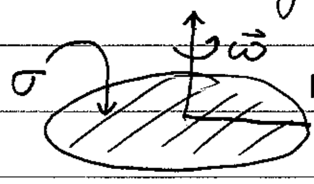
When $z \gg R$, we expect the field to be predominantly dipolar:

$$z \gg R \Rightarrow \vec{B}(z) \approx \frac{\mu_0}{4\pi} \left[\frac{2\pi R^2 I \, \hat{z}}{z^3} \right]$$

$$= \frac{\mu_0}{4\pi} \left(\frac{2m\hat{z}}{z^3} \right)$$

As expected, since on z-axis $\hat{r} = \hat{z}$, $r = z$, $\theta = 0$

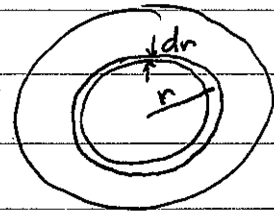
Example: Rotating disk of Charge



$$\vec{K} = \sigma v(r) = \sigma \omega r \,\hat{\phi}$$

(surface current density)

Let us find the magnetic dipole moment.
To do so we will break up the disk into rings (loops)



$$dI(r) = K(r)\, dr$$
$$= \sigma \omega r\, dr$$
$$= \frac{2\pi r\, dr\, \sigma}{T} \quad \left(\text{where } \omega = \frac{2\pi}{T}\right)$$
$$= \text{Charge/Time} \checkmark$$

$\Rightarrow$ Magnetic dipole/ring

$$dm(r) = \pi r^2\, dI(r) = (\pi \sigma \omega)\, r^3\, dr$$

The total magnetic dipole moment

$$\boxed{m = \int_0^R dm(r) = \frac{1}{4}\pi \sigma \omega R^4}$$

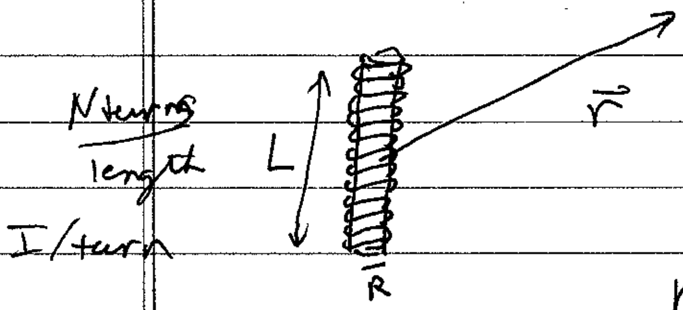
From lecture 24, we found (via Biot-Savart), the exact $\vec{B}$-field on z-axis

$$\vec{B}(z) = \frac{\mu_0 \sigma \omega}{2}\left[-2z + \frac{R^2 + 2z^2}{\sqrt{R^2 + z^2}}\right]\hat{z}$$

$$\lim_{z \gg R} \vec{B}(z) \approx \frac{\mu_0}{8}\frac{\sigma \omega R^4}{z^3}\hat{z} = \left(\frac{\mu_0}{4\pi}\right)\frac{2m}{z^3}\hat{z} \checkmark$$

Pseudo-Charge "Gilbert Model"

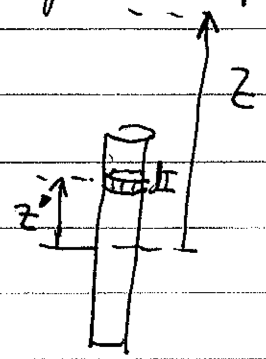
Consider a finite solenoid:



when $|\vec{r}| \gg L$, this looks like a magnetic dipole

$$m = (\#\text{turns})\, I A = N L I \pi R^2$$

Consider B-field on z-axis: Supperposition of magnetic dipole fields



$$dB(z) = \frac{\mu_0}{4\pi}\left(\frac{2\,dm}{(z-z')^3}\right) = \frac{\mu_0}{4\pi}\left(2\,\frac{N dz' I A}{(z-z')^3}\right)$$

$$\Rightarrow B(z) = \int_{-L/2}^{L/2} dB(z) = \frac{\mu_0}{4\pi} N I A \int_{-L/2}^{L/2} \frac{2dz'}{(z-z')^3}$$

$z \gg R$ (not necessarily L) $\Rightarrow B(z) = \frac{\mu_0}{4\pi} N I A \left(\frac{1}{(z-L/2)^2} - \frac{1}{(z+L/2)^2}\right)$

Looks like the electric field of two point charges

Define "pole strength" $q^* = NIA = \frac{m_{total}}{L}$

$$\Rightarrow m_{total} = q^* L \quad \text{(Like electric dipole)}$$

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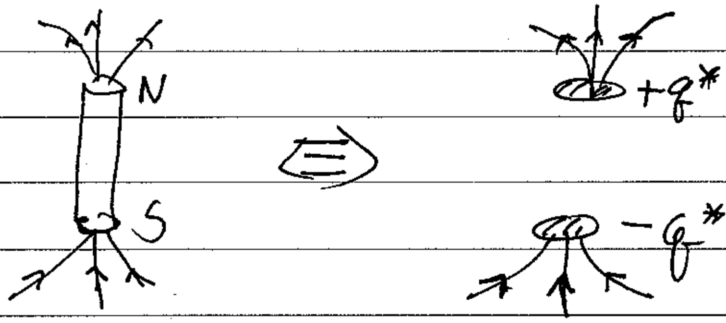
$$B(z) \cong \frac{\mu_0}{4\pi} \left(\frac{q^*}{(z-L/2)^2} - \frac{q}{(z+L/2)} \right)$$



We can thus model a magnetic dipole as two oppositely charged "magnetic monopoles", bound together over ~~the~~ length l, s.t.

$$q^* = m/l$$

<u>Aside:</u> The Gilbert model is intimately related to the following mathematical fact. Suppose we are in a region where $\vec{J} = 0$ (outside current)

Then $\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla} U_m$

where U_m is a magnetic "pseudo potential".

"Pseudo" because this cannot be true everywhere (unless $U = 0$ $\vec{B} = 0$)

$\Rightarrow$ We can mimick the magnetic field as being created by "pseudo charges"

(more later)