Magneto-Statics in Matter

Ampère's Law in Magnetizable Material

In the same way that we studied electric fields in the presence of polarizable (or polarized) materials, we want to study magnetic fields in the presence of magnetizable (or magnetized) materials.

In lecture 28 we showed that the magnetic field associated with magnetization \( \vec{M}(\hat{n}) \) can we understood as arising from "bound current" \( \vec{J}_b = \vec{\nabla} \times \vec{M} \) (on the surface of the material, this gives \( \vec{K}_b = \vec{M} \times \hat{n} \)).

The total current density is then the "free current" (flowing in the conductors), plus the "bound current" that is associated with the magnetization.

Eq. "Electromagnet" \( \hat{n} \) "free"

\( \hat{n} \) "magnetizable"
In contrast, in electrostatics, we typically do not set \( Q_{\text{free}} \), but instead we set \( V \).

Eq. Capacitor

\[
\begin{array}{c}
\text{V} \\
\text{c} \\
\text{Q} = CV
\end{array}
\]

Thus, especially in electrical engineering context, the quantities of interest are

\[
\begin{align*}
V & \leftrightarrow E \\
I & \leftrightarrow H
\end{align*}
\]

The quantities \( B \) and \( D \) are considered "auxiliary." In fact, in many texts, one call \( H \) "the magnetic field" and \( B \) the "magnetic flux density" or "magnetic induction" because of its relation to Faraday's law (to be discussed next semester). In physics, though, \( B \) is a fundamental physical quantity.

Note: Mathematically there is a closer relation between

\[
\begin{align*}
\vec{E} & \leftrightarrow \vec{H} \\
\vec{D} & \leftrightarrow \vec{B}
\end{align*}
\]

\[
\begin{align*}
\vec{B} & = \varepsilon_0 \vec{E} + \vec{P} \\
\vec{B} & = \mu_0 \vec{H} + \vec{M}
\end{align*}
\]
Total current density: $\mathbf{J} = \mathbf{J}_{\text{total}} = \mathbf{J}_{\text{bound}} + \mathbf{J}_{\text{free}}$

Ampère's Law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{total}} = \mu_0 (\mathbf{J}_{\text{bound}} + \mathbf{J}_{\text{free}})$

$\Rightarrow \nabla \times (\frac{1}{\mu_0} \mathbf{B}) - \mathbf{J}_{\text{bound}} = \mathbf{J}_{\text{free}}$

$\Rightarrow \nabla \times (\frac{1}{\mu_0} \mathbf{B} - \mathbf{M}) = \mathbf{J}_{\text{free}}$

$\Rightarrow \mathbf{H} = \mathbf{J}_{\text{free}}$

$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$

$\mathbf{H} = "\text{Auxiliary Magnetic Field}"$

$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}}$

$\mathbf{H}$ plays the role $\mathbf{D}$ did in electrostatics.

Note: $\mathbf{H}$ is a much more useful quantity than $\mathbf{D}$ since in the laboratory we set the free current (next page)
Pseudo-charge density for magnetized material

Suppose there is no free current, only bound (e.g., permanent magnet).

\[
\nabla \times \vec{H} = 0
\]
\[
\nabla \cdot \vec{H} = \nabla \cdot \left( \frac{1}{\mu_0} \vec{B} - \vec{M} \right)
\]
\[
\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}
\]

Thus, \( \vec{H} \) satisfies the same field equations as the electrostatic field.

\[
\nabla \times \vec{H} = 0 \Rightarrow \vec{H} = -\nabla V_M
\]
\[
\nabla \cdot \vec{H} = -\frac{\text{Pseudo bound charge}}{\varepsilon_0} \quad \text{density (Gilbert model)}
\]

\[
\nabla \cdot \vec{M} = \text{pseudo charge density (in bulk)}
\]
\[
\vec{H} \cdot \vec{M} = \text{pseudo surface charge density}
\]
Example: Magnetized solids

- Bar Magnet:

\[ \mathbf{H} \quad \Rightarrow \quad \mathbf{B} \]

\[ \text{Source for } \mathbf{B} \quad \text{Source for } \mathbf{H} \]

\[ \mu_0 \mathbf{H} \]

\[ \mathbf{M} \quad \mathbf{B} \]

- Magnetized Sphere

\[ \mathbf{M} \quad \Rightarrow \quad \mathbf{K} \]

\[ \text{Source for } \mathbf{B} \quad \text{Source for } \mathbf{H} \]

\[ \mathbf{O}^* (\theta) = \mathbf{n}, \quad \mathbf{H} = \mathbf{M} \cos \theta \Rightarrow \mathbf{H} \text{ for magnetized sphere} \]

is the same as \( \mathbf{E} \) for electrically polarized sphere:

\[ \mathbf{M} \quad \mathbf{H} \]

\[ \mathbf{B} \]
Linear response

In paramagnetic and diamagnetic materials, for "reasonable strength" fields, the magnetization is proportional to the applied field.

By convention, we define a "magnetic susceptibility" in terms of \( \mathbf{H} \):

\[
\mathbf{M} = X_m \mathbf{H} \quad \text{(linear response)}
\]

\[
\rightarrow \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 (1 + X_m) \mathbf{H} = \mu \mathbf{H} = \mu
\]

\[
\mathbf{M} = \mu_0 (1 + X_m) \equiv \text{Magnetic permeability}
\]

When the material response is linear, we can solve for \( \mathbf{B} \) in the presence of \( \mu \) in the same way that we solved for \( \mathbf{E} \) in the presence of \( \mathbf{E} \). The magnetic permeability accounts for all the effects of bound current (or equivalent pseudo charge).
Example: Solenoid filled with "μ-metal"

\[ H = NI \hat{z} \]
\[ \Rightarrow \overrightarrow{B} = \mu_N I \hat{z} \quad \text{(increased)} \]

Example: Coaxial cable filled with non-conducting "μ-material"

\[ \mu \rightarrow I \quad \phi \]

Between cables: \[ H = \frac{I}{2\pi r} \hat{\phi} \]
\[ \Rightarrow \overrightarrow{B} = \frac{\mu I}{2\pi r} \hat{\phi} \]

Magnetic Shielding

Unlike for electric fields, there are no conductors of magnetic charges. However, a very magnetized sample is equivalent to a highly polarized sample with pseudo-charge

"μ-metal" shielding
Boundary Conditions

Maxwell's Equations for "macroscopic" magnetic fields:

\[ \nabla \times \mathbf{H} = \mathbf{J}_\text{free} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \mathbf{H} > - \mathbf{H} < = K_\text{free} \times \mathbf{n} \]

\[ B_n^> - B_n^< = 0 \]

In the absence of free current, \( H_\parallel \) is continuous.

\( B_n \) is always continuous.

Note: \( \mathbf{B} > - \mathbf{B} < = \mu_0 \mathbf{K}_\text{total} \times \mathbf{n} \)

\[ \mathbf{B} > - \mathbf{B} < = \mu_0 (K_\text{free} + K_\text{bound}) \times \mathbf{n} \]

Example: Solenoid filled with "\( \mu \)-material".

\[ \mathbf{H}_{\text{free}} = \mathbf{n} I = K_\text{free, inside} \]

\[ 0 \text{ outside} \]

\[ \mathbf{H}_\parallel = 0 \text{ everywhere} \]

\[ \Delta H_\parallel = K_\text{free} \checkmark \]

(Next Page)
\[ B_{\|} \text{ in} = \mu_0 n I = \mu K^f = \mu_0 (1 + \chi_m) K^f \]

\[ = \mu_0 |K^f| \left| \frac{1}{K^f_{\text{free}}} + \mu_0 \chi_m |K^f_{\text{free}}| \right| \]

\[ = \mu_0 |K^f_{\text{free}}| \frac{\chi_m |\mathbf{H}_{\|}|}{|\mathbf{M}_{\times \hat{n}}|} \]

\[ B_{\|} = \mu_0 |K^f_{\text{free}}| + \mu_0 |K^f_{\text{bound}}| \]