Physics 405: Lecture 29

Magneto-Statics in Matter

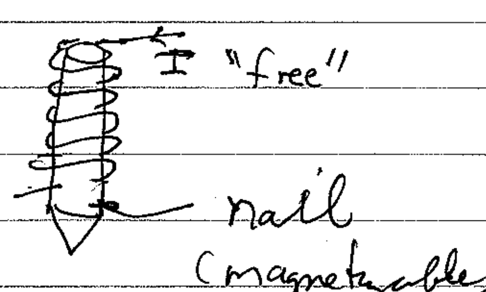
Ampère's Law in Magnetizable Material

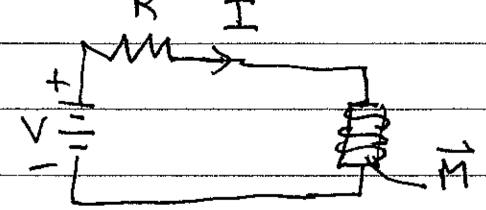
In the same way that we studied electric fields in the presense of polarizable (or polarized) materials, we want to study magnetic fields in the presense of magnetizable (or magnetized) materials.

In Lecture 28 we showed that the magnetic field associated with magnetization $\vec{M}(\vec{r})$ can we understood as arising from "bound current" $\vec{J}_b = \vec{\nabla}\times\vec{M}$ (on the surface of the material, this gives $\vec{K}_b = \vec{M}\times\hat{n}$)

The total current density is then the "free current" (flowing in the conductors), plus the "bound current" that is associated with the magnetization

Eg. "Electromagnet"

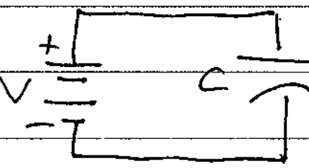




$$I = V/R$$

In contrast, in electrostatics, we typically do not set Q_{free}, but instead we set $\underline{V}$

Eq. Capacitor



$Q = CV$

Thus, especially in electrical engineering context, the quantities of interest are

$$V \Leftrightarrow E$$

$$I \Leftrightarrow H$$

The quantities $\vec{B}$ and $\vec{D}$ are considered "auxilary"
In fact in many texts, one call $\vec{H}$ "the magnetic field" and $\vec{B}$ the "magnetic flux density" or "magnetic induction" because of its relation to Faradays Law (to be discussed next semester). In physics though, $\vec{B}$ is the <u>fundamental</u> physical quantity.

<u>Note</u>: Mathematical there is a closer relation between

$$\vec{E} \Leftrightarrow \vec{H} \qquad \vec{D} \Leftrightarrow \vec{B}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{B} = \mu_0 \vec{H} + \vec{M}$$

Total current density: $\vec{J}_{total} = \vec{J}_{bound} + \vec{J}_{free}$

Ampère's Law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{total} = \mu_0 (\vec{J}_{bound} + \vec{J}_{free})$

$$\Rightarrow \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B}\right) - \underbrace{\vec{J}_{bound}}_{= \vec{\nabla} \times \vec{M}} = \vec{J}_{free}$$

$$\Rightarrow \vec{\nabla} \times \underbrace{\left(\frac{1}{\mu_0} \vec{B} - \vec{M}\right)}_{\equiv \vec{H}} = \vec{J}_{free}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{\nabla} \times \vec{H} &= \vec{J}_{free} \\ \vec{H} &= \frac{1}{\mu_0} \vec{B} - \vec{M} \end{aligned}}$$

$\vec{H}$ = "Auxilary Magnetic Field"

$$\boxed{\oint \vec{H} \cdot d\vec{l} = I_{free}^{enclosed}}$$

$\vec{H}$ plays the role ~~of~~ $\vec{D}$ did in electrostatics.

<u>Note</u>: $\vec{H}$ is a much more useful quantity than $\vec{D}$ since in the laboratory we <u>set</u> the free current (next page)

Pseudo-Charge density for Magnetized material

Suppose there in no free current, only bound (e.g. permanent magnet)

Here $\boxed{\vec{\nabla}\times\vec{H} = 0}$

and $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right)$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}}$$

Thus, $\vec{H}$ satisfies the same field equations as the electrostatic field.

$$\vec{\nabla}\times\vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla}V_M$$

$$\vec{\nabla}\cdot\vec{M} = -\frac{\rho^*_{bound}}{\epsilon_0} \leftarrow \text{Pseudo bound charge density (Gilbert model)}$$

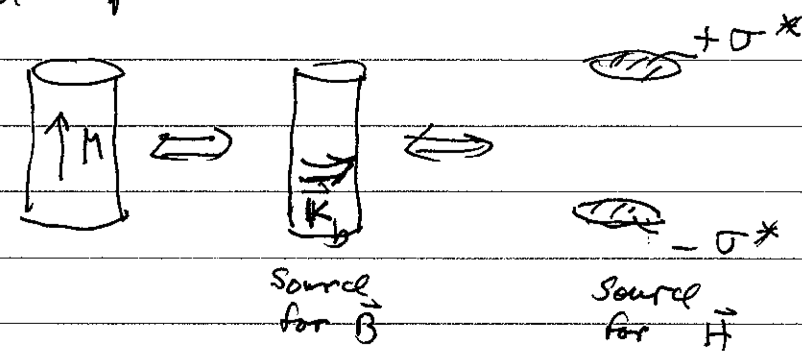
$$\Rightarrow V_M = \frac{1}{4\pi}\int\left(\frac{-\vec{\nabla}\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d^3r'\right)$$

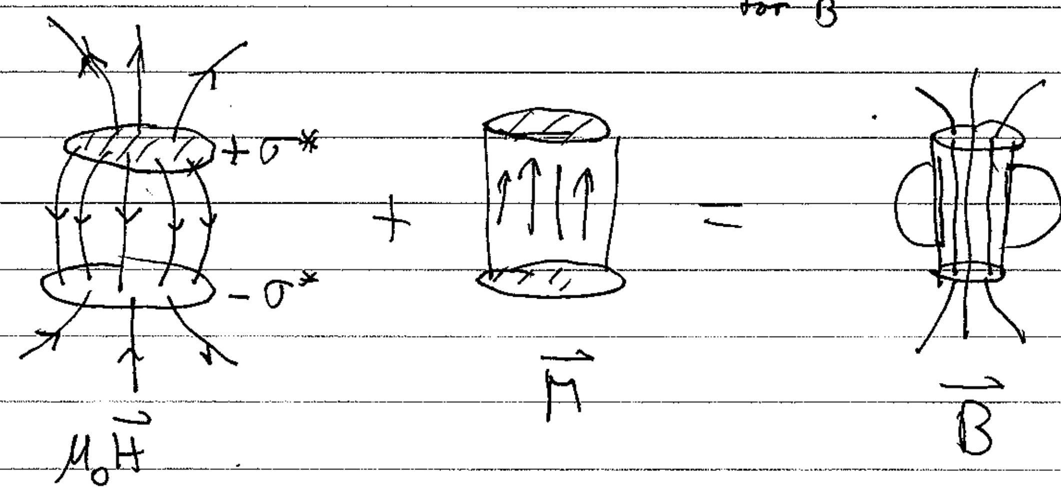
$-\vec{\nabla}\cdot\vec{M}$ = pseudo charge density (in bulk)

$\hat{n}\cdot\vec{M}$ = pseudo surface charge density

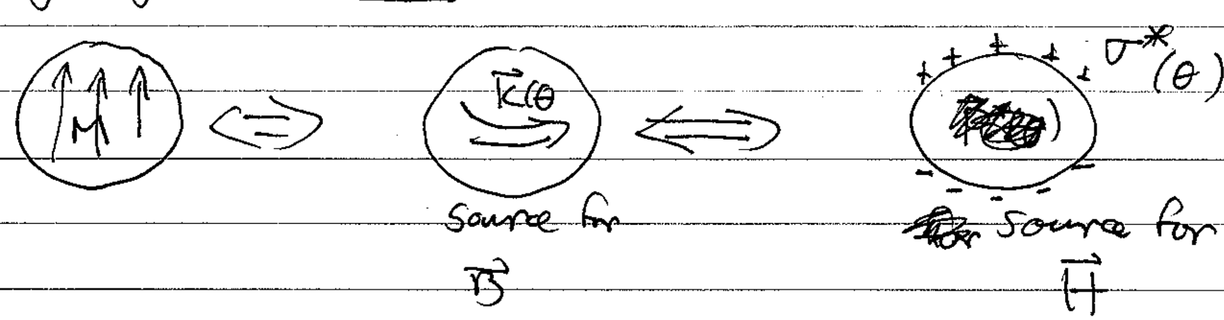
Example: Magnetized ~~s~~ solids

- **Bar Magnet:**

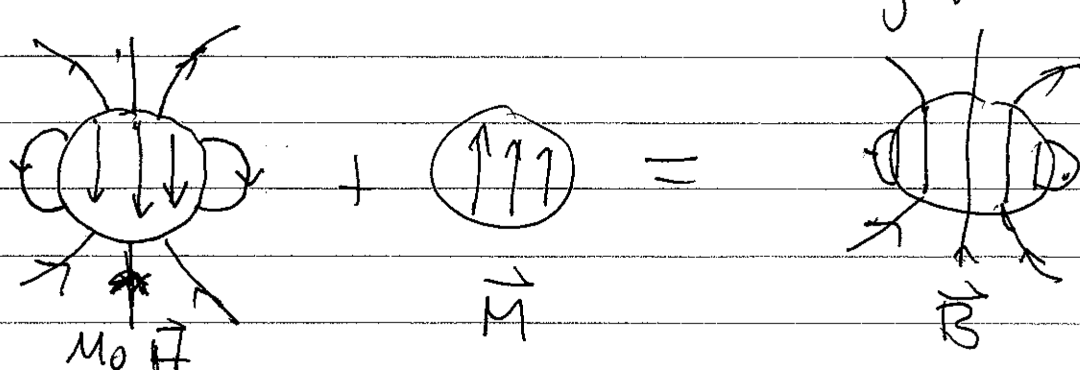




- **Magnetized Sphere**



$\sigma^*(\theta) = \hat{n}\cdot\vec{M} = M\cos\theta \Rightarrow \vec{H}$ for magnetized sphere is the same as $\vec{E}$ for electrically polarized sphere.



Linear response

In paramagnetic and ~~diag~~ diamagnetic materials, for "reasonable strength" fields, the induced magnetization in proportional the applied field.

By convention, we define a "magnetic susceptibility" in terms of $\vec{H}$

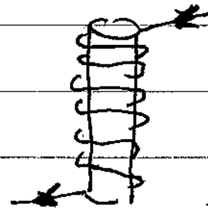
$$\vec{M} = \chi_m \vec{H} \quad \text{(Linear reponse)}$$

$$\Rightarrow \vec{B} = \mu_0(\vec{H} + \vec{M}) = \underbrace{\mu_0(1+\chi_m)}_{\equiv \mu} \vec{H} = \mu \vec{H}$$

$$\boxed{\mu = \mu_0(1+\chi_m) \equiv \text{Magnetic permeability}}$$

When the material response is linear, we can solve for $\vec{B}$ in the presence of μ In the same way that we solved for $\vec{E}$ in the presence of ϵ. The magnetic permeability accounts for all the effects of bound current (or equivalent pseudo charge).

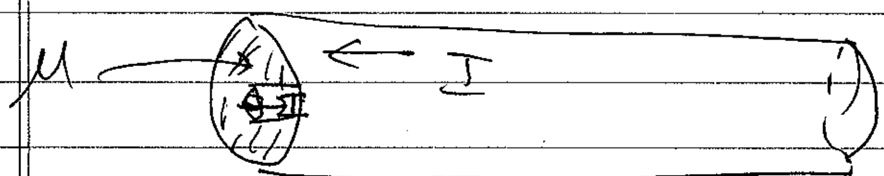
Example: Solenoid filled with "μ-material"



$$\vec{H} = NI\,\hat{z}$$

$$\Rightarrow \vec{B} = \mu N I\,\hat{z} \quad \text{(increased)}$$

Example: Coaxial cable filled with non-conducting "μ-material"

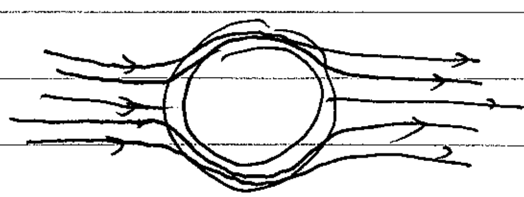


Between cables: $\vec{H} = \dfrac{I}{2\pi r}\,\hat{\phi}$

$$\Rightarrow \vec{B} = \frac{\mu I}{2\pi r}\,\hat{\phi}$$

Magnetic Shielding

Unlike for electric fields, there are no conductors of magnetic charges. However, a very magnetized sample is equivalent to a highly polarized sample with pseudo-charge

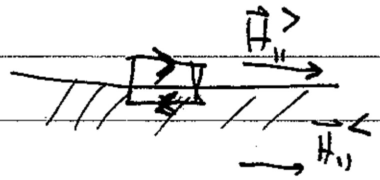


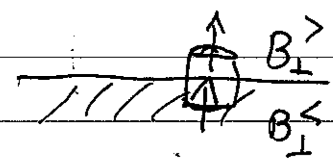
"μ-metal" shielding

Boundary Conditions

Maxwell's Eqns. for "macroscopic" magnetic fields

$$\underbrace{\vec{\nabla}\times\vec{H} = \vec{J}_{free}}_{\Downarrow} \qquad\qquad \underbrace{\vec{\nabla}\cdot\vec{B} = 0}_{\Downarrow}$$





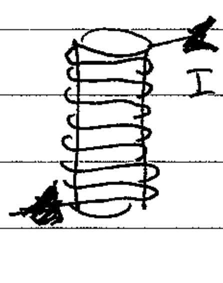
$$\Rightarrow \vec{H}_{\parallel}^{>} - \vec{H}_{\parallel}^{<} = \vec{K}_{free}\times\hat{n} \qquad\qquad B_n^{>} - B_n^{<} = 0$$

In the absence of free current, $H_{\parallel}$ is continuous.
B_n is <u>always</u> continuous

Note: $\vec{B}_{\parallel}^{>} - \vec{B}_{\parallel}^{<} = \mu_0 \vec{K}_{total}\times\hat{n}$

$$\Rightarrow \vec{B}_{\parallel}^{>} - \vec{B}_{\parallel}^{<} = \mu_0(\vec{K}_{free} + \vec{K}_{bound})\times\hat{n}$$

<u>Example</u>: Solenoid filled with "μ-material"



$$\vec{H}_{\parallel} = \begin{cases} nI = K_{free} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

$$\vec{H}_{\perp} = 0 \text{ everywhere}$$

$\Rightarrow \Delta H_{\parallel} = K_{free}$ ✓

(Next Page)

$$B_{\parallel} \text{ in } = \mu n I = \mu K_{free} = \mu_0 (1+\chi_m) K_{free}$$

$$= \mu_0 |\vec{K}_{free}| + \mu_0 \chi_m |\vec{K}_{free}|$$

$$= \mu_0 |\vec{K}_{free}| + \mu_0 \underbrace{\chi_m |\vec{H}_{\parallel}|}_{|\vec{M} \times \hat{n}|}$$

$$B_{\parallel} = \mu_0 |\vec{K}_{free}| + \mu_0 |\vec{K}_{bound}| \checkmark$$