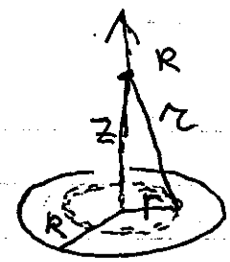
Physics 405

P.S. #3

Solutions

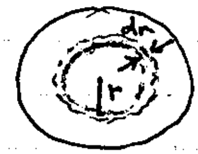
Prob. 1

(b) Flat circular disc, radius R, surface charge σ



Break up the disc into rings of radius r and differential thickness dr

"Top" view



The charge/length in the differential ring is

$$d\lambda = \frac{\sigma\,(\text{Area ring})}{(\text{circumference})} = \sigma\, dr$$

Each ring contributes a electric field given in part a

$$dE_z = \frac{d\lambda}{2\epsilon_0}\frac{r z}{(r^2+z^2)^{3/2}} = \frac{\sigma}{2\epsilon_0}\, z\, \frac{r\,dr}{(r^2+z^2)^{3/2}}$$

Integrating over the area of the disc $(0 < r \leq R)$

$$E_z = \frac{\sigma}{2\epsilon_0}\, z \int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}} = \frac{\sigma}{2\epsilon_0}\, z \left[-(r^2+z^2)^{-1/2}\right]_0^R$$

$$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\, z \left(\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{R^2+z^2}}\right)\hat{z}}$$

In the limit $R \to \infty$; $z/R \ll 1$

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\frac{z}{\sqrt{z^2}}\left(1 - \left(\frac{\frac{z^2}{R^2}}{1 + z^2/R^2}\right)^{1/2}\right)\hat{z} \qquad \left(\begin{array}{c}\text{Dimensionless}\\ \text{Form}\end{array}\right)$$

To lowest nonvanishing order

$$\vec{E} \cong \frac{\sigma}{2\epsilon_0}\frac{z}{\sqrt{z^2}}\hat{z} = \frac{\sigma}{2\epsilon_0}\,\text{sign}(z)\,\hat{z}$$ Field of an infinite plane ✓

(1b) continued

Now let's take the other limit (this is trickier)

limit $z >> R \Rightarrow \frac{R}{z} << 1$

Our small parameter is $\delta \equiv \frac{R}{z}$. The first trick is dimensionless form

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \frac{z}{\sqrt{z^2}} \left(1 - \sqrt{\frac{z^2}{R^2+z^2}}\right)\hat{z} = \frac{\sigma}{2\epsilon_0} \frac{z}{\sqrt{z^2}} \left(1 - \frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\right)\hat{z}$$

$$= \frac{\sigma}{2\epsilon_0} \frac{z}{\sqrt{z^2}} \left(1 - (1+\delta^2)^{-1/2}\right) \hat{z} \qquad \text{(So far \underline{exact})}$$

Now use our favorite: binominal expansion

$$(1+\delta^2)^{-1/2} \simeq 1 - \tfrac{1}{2}\delta^2$$

$$\Rightarrow \vec{E} \approx \frac{\sigma}{2\epsilon_0} \frac{z}{\sqrt{z^2}} \left(1 - (1 - \tfrac{1}{2}\delta^2)\right)\hat{z} = \frac{\sigma}{2\epsilon_0} \frac{z}{\sqrt{z^2}} \left(\tfrac{1}{2}\delta^2\right)\hat{z}$$

$$\Rightarrow \vec{E} \simeq \frac{\sigma}{4\epsilon_0} \frac{z}{\sqrt{z^2}} \frac{R^2}{z^2}\hat{z} = \frac{1}{4\pi\epsilon_0} \frac{\sigma\pi R^2}{z^2} \hat{z}$$

But $\sigma\pi R^2 = Q_{total}$ (the total charge)

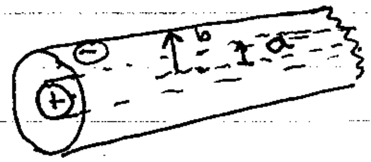
So on the z-axis with $z >> R$

$$\boxed{\vec{E} \simeq \frac{1}{4\pi\epsilon_0} \frac{Q_{total}}{z^2} \hat{z}}$$

As expected: the field of a point charge which $\vec{r} = z\hat{z}$ the distance from charge to observer

Problem ~~3~~

Griffiths 2.16, page ~~58~~ 77



Find electric field. Symmetry $\Longrightarrow$ Use Gauss' Law

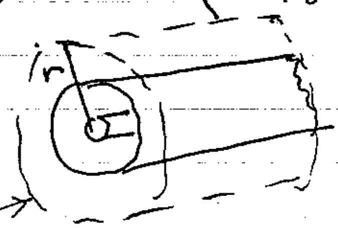
Take "Gaussian surface" to be a cylinder

The E-field must have radial symmetry

(i.e. $\vec{E}(\vec{r}) = E(r)\,\hat{r}$) $\Longrightarrow$ ~~$E(\rho)\,2\pi\rho L = \frac{Q_{enc}}{\epsilon_0}$~~

cylindr. coords

(I) $r > b$ (outside the big cylinder)



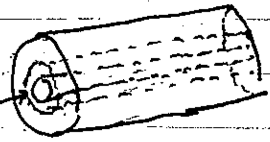
Gaussian surface

Since there are equal and opposite charges distributed on the inner + outer cylinder the total enclosed charge is zero

$$\Rightarrow E(r)\,2\pi r L = \frac{Q_{enc}}{\epsilon_0} = 0$$

$$\Rightarrow \boxed{E(r) = 0 \qquad r > b}$$

(II) $a < r\ b$ (in between cylinders)



Gaussian surface

$$E(r)\,2\pi r L = \frac{Q_{enc}}{\epsilon_0}$$

$Q_{enclosed} = \rho A L$ where A is the cross section of little cyl.

$= \rho \pi a^2 L$

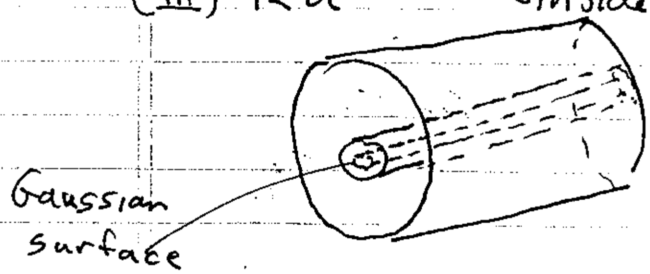
$$\therefore\ E(r)\,2\pi r L = \frac{\pi a^2 L}{\epsilon_0}\rho$$

Field of line charge

$$a<r<b \Longrightarrow \boxed{E(r) = \frac{a^2}{2\epsilon_0}\frac{\rho}{r} = \frac{2\lambda}{4\pi\epsilon_0 r}, \quad \lambda = \pi a^2 \rho = \frac{\text{charge}}{\text{length}}}$$

Problem 3 continued

(III) $r<a$ (inside little cylinder)



$$E(r)\, 2\pi r L = \frac{Q_{enc}}{\epsilon_0} = \int_V dV \frac{\rho}{\epsilon_0}$$

To find Q_{enc} we must integrate the charge density ρ over the volume enclosed by the Gaussian surface

$$Q_{enclosed} = \int_{G.S.} dV \rho = \rho \int dV = \rho \underbrace{\pi r^2 L}_{\text{Volume of Gaussian surface}}$$

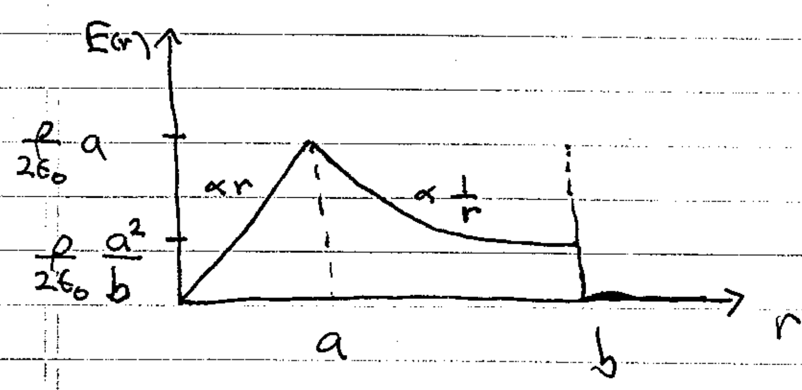
$$\Rightarrow \quad E(r)\, 2\pi r L = \frac{\rho}{\epsilon_0} \pi r^2 L$$

$$\Rightarrow \quad E(r) = \frac{\rho}{2\epsilon_0}$$

In summary

$$\vec{E}(\vec{r}) = E(r)\,\hat{r}, \qquad E(r) = \begin{cases} 0 & r>b \\ \frac{\rho}{2\epsilon_0}\frac{a^2}{r} & a<r<b \\ \frac{\rho}{2\epsilon_0} r & r<a \end{cases}$$

Plot $|\vec{E}(\vec{r})| = E(r)$

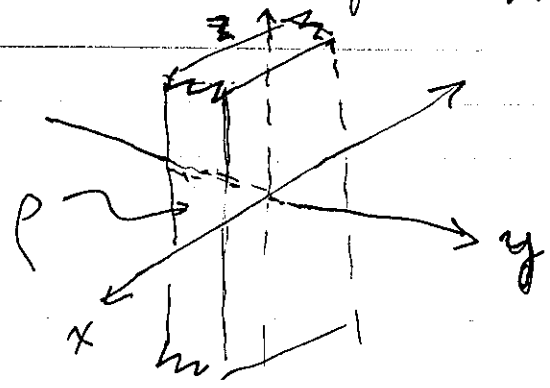


Note: there is a discontinuity in $E(r)$ a $r=b$
This is due to the surface charge.

$$E(r=b_-) - E(r=b_+) = \frac{\sigma}{2\epsilon_0}$$

Problem 3: Griffiths: Problem 2.17

An infinite slab of thickness 2d, volume density ρ



By symmetry, the field must point in the y-direction

$\Rightarrow \quad \vec{E} = E(y)\,\hat{y}$

Also, there is reflection symmetry. That is

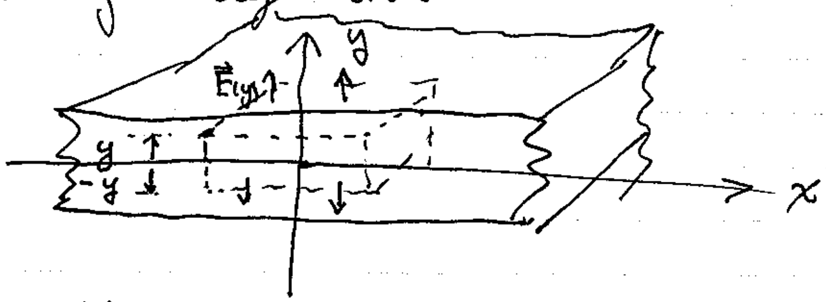
$|\vec{E}(y)| = |\vec{E}(-y)|$ (the charge looks the same on either side)

And $E(y) = -E(-y)$ (points in opposite direction)

We must consider three regions

(I) $-d < z < d$, (II) $z < -d$, (III) $z > d$

(I) Looking "edge-on":



Inside the slab I have draw a "pill box" with flat top at y bottom at $-y$

Since $\vec{E}$ points in the $\hat{y}$ direction there is flux only through the top and bottom of the pill box

$$\oint \vec{E}\cdot d\vec{a} = |\vec{E}_{top}|\,A_{top} + |\vec{E}_{bottom}|\,A_{bottom}$$

$$|\vec{E}_{top}| = |\vec{E}_{bottom}| = E(y)$$

$$A_{top} = A_{bottom} \equiv A \quad \text{(arbitrary area)}$$

- Region (I) continued $-d < y < d$

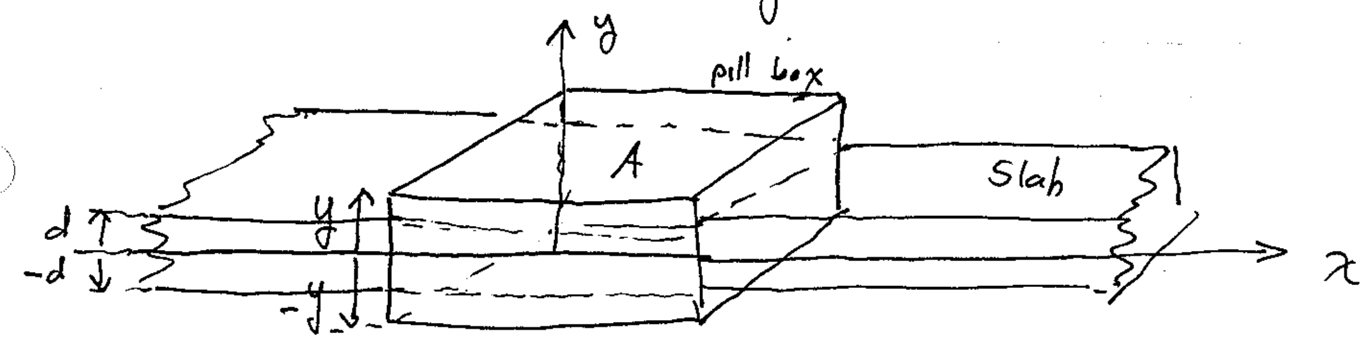
$$\underset{\text{"pill box"}}{\oint} \vec{E} \cdot d\vec{a} = E(y)\,(2A) = \frac{Q_{enc}}{\epsilon_0}$$

$$Q_{enc} = \rho\left(V_{pill\text{-}box}\right) = \rho\, A\,(2y)$$

$$\therefore \quad -d < y < d \qquad \boxed{E(y) = \rho/\epsilon_0\, y}$$

- Region II and III $\quad |y| > d$

Take pill box straddling the slab



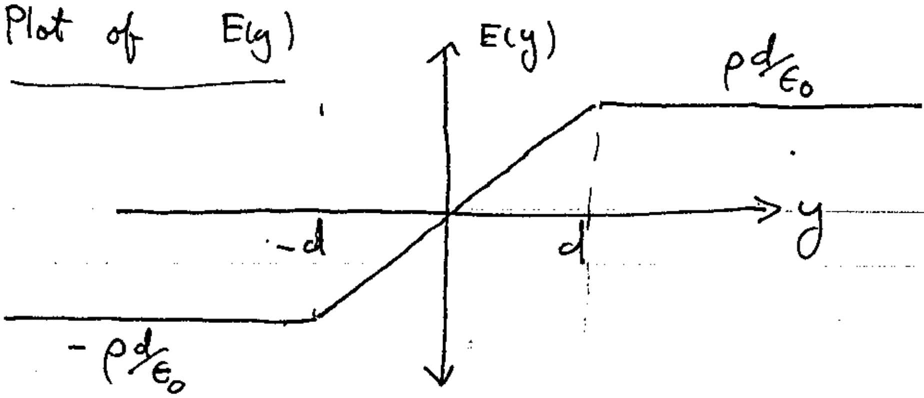
Again $\oint \vec{E} \cdot d\vec{a} = |\vec{E}(y)|\, 2A = Q_{enc}/\epsilon_0$

But $Q_{enc} = \rho\, A\,(2d)$ (enclosed charge independent of y)

$$\Rightarrow \quad |y| > d \qquad \boxed{E(y) = \rho/\epsilon_0\, d}$$

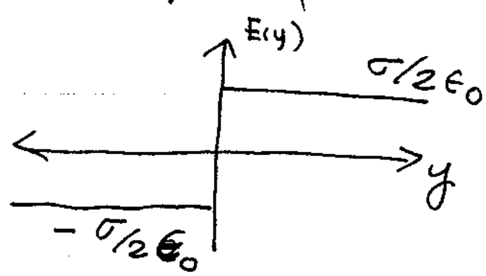
(Next Page)

Plot of $E(y)$



Grows linearly for $-d<y<d$
constant otherwise

In the limit $d \to 0$, so all the charge is concentrated on a plane, $2\rho d \to \sigma$ (charge/area)



So at the interface there is a discontinuity in electric field

$$\boxed{\Delta E\big|_{surface} = E_{above} - E_{below} = \frac{\sigma}{\epsilon_0}}$$

This is general result for electric fields normal to surfaces which have surface-charge

Eg. In problem 3, there is a discontinuity in $|\vec{E}|$ at $r=b$

$$E(r=b_-) - E(r=b_+) = \frac{\rho a^2}{2\epsilon_0 b}$$

The surface charge at $r=b$ was supposed to cancel the charge of the inner cylinder

$$2\pi b\sigma = \rho\pi a^2 \Rightarrow \sigma = \frac{\rho a^2}{2b} \quad \therefore \quad \Delta E\big|_b = \frac{\sigma}{\epsilon_0} \checkmark$$

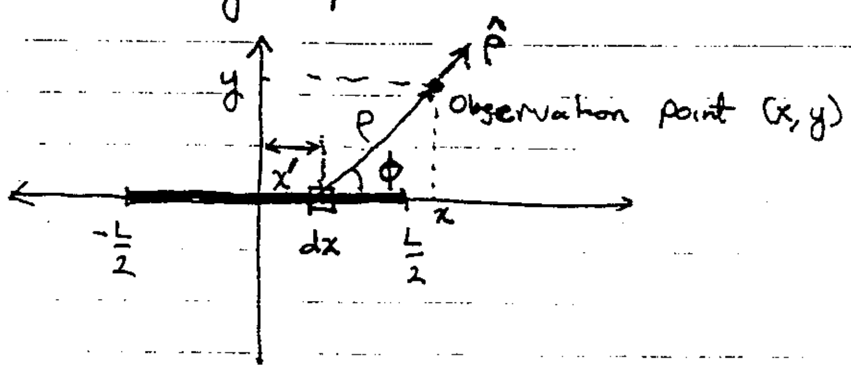
(4) An infinite strip of charge per unit area σ.
Find: $\vec{E}(x,y,z)$

(a) The key to this problem is the superposition principle

(i) By <u>symmetry</u> the electric field cannot have an $\hat{z}$ component since it is infinite in this direction. Thus we can evaluate $\vec{E}$ solely in the x-y plane

Now, given the hint in the assignment, we break up the strip into infinitesimal strips of width dx. Each one of the infinitesimal strips represents a <u>line charge</u> with $d\lambda = \sigma dx$ charge per unit length

Look in x-y plane with the <u>out of page</u>



The electric field due to the line charge at at x' at the observation point (x,y) is

$$d\vec{E}(x,y) = \frac{1}{4\pi\epsilon_0}\left(\frac{2\,d\lambda}{\rho}\right)\hat{\rho} \quad \text{(Field of a line charge)}$$

By geometry $\rho = \sqrt{(x-x')^2+y^2}$ (go Pythagorus!)

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}, \quad \cos\phi = \frac{x-x'}{\rho}, \quad \sin\phi = y/\rho$$

4 continued

Thus, the field due to the infinitesimal strip is

$$d\vec{E} = \frac{\sigma}{2\pi\epsilon_0}\left\{\frac{dx'}{\rho}\left(\frac{x-x'}{\rho}\hat{x} + \frac{y}{\rho}\hat{y}\right)\right\}$$

$$= \frac{\sigma}{2\pi\epsilon_0}\left(\frac{x-x'}{[(x-x')^2+y^2]^{3/2}}\hat{x} + \frac{y}{[(x-x')^2+y^2]^{3/2}}\hat{y}\right)dx'$$

Note: $d\vec{E}$ depends on both the observation point (x,y) and the position of the ~~[illegible]~~ infinitesimal strip, x'.

Now we fix the observation point and add up (i.e. integrate) over all the line charges

$$\vec{E}(x,y) = \int d\vec{E} = \frac{\sigma}{2\pi\epsilon_0}\int_{-L/2}^{L/2} dx'\left\{\frac{x-x'}{[(x-x')^2+y^2]}\hat{x} + \frac{y}{[(x-x')^2+y^2]}\hat{y}\right\}$$

OK, so now we have to do ~~some~~ integration

Let $I_1 \equiv \int_{-L/2}^{L/2} dx' \frac{x-x'}{[(x-x')^2+y^2]}$

$$I_2 \equiv \int_{-L/2}^{L/2} dx' \frac{y}{[(x-x')^2+y^2]} = \int_{-L/2}^{L/2}\frac{dx'}{y}\frac{1}{\left(\frac{x-x'}{y}\right)^2+1}$$

$$\Rightarrow \vec{E}(x,y) = \frac{\sigma}{2\pi\epsilon_0}\left(I_1(x,y)\,\hat{x} + I_2(x,y)\,\hat{y}\right)$$

2f continued

Let $u_1 = (x-x')^2 + y^2 \Rightarrow du_1 = 2(x-x')(-dx')$ (Remember, x, y are fixed)

$$\therefore\ I_1(x,y) = -\frac{1}{2}\int_{(x+\frac{L}{2})^2+y^2}^{(x-\frac{L}{2})^2+dy^2} \frac{du_1}{u_1} = -\frac{1}{2}\ln(u_1)\Big|_{(x+\frac{L}{2})+y^2}^{(x-\frac{L}{2})^2+y^2}$$

$$= \frac{1}{2}\left[\ln\left[(x+\tfrac{L}{2})^2+y^2\right] - \ln\left[(x-\tfrac{L}{2})^2+y^2\right]\right]$$

$$\Rightarrow\ I_1(x,y) = \frac{1}{2}\ln\left[\frac{(x+\frac{L}{2})^2+y^2}{(x-\frac{L}{2})+y^2}\right],$$ having used $\ln(A)-\ln(B) = \ln\left(\frac{A}{B}\right)$.

Now let $u_2 = \frac{x-x'}{y} \Rightarrow du_2 = -\frac{dx'}{y}$

$$\therefore\ I_2(x,y) = -\int_{\frac{x+L/2}{y}}^{\frac{x-L/2}{y}} \frac{du_2}{1+(u_2)^2} = -\tan^{-1}(u_2)\Big|_{\frac{x+L/2}{y}}^{\frac{x-L/2}{y}}$$

$$\Rightarrow\ I_2(x,y) = \tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)$$

Putting this all together we get the desired result

$$\boxed{\vec{E}(\vec{r}) = \frac{\sigma}{2\pi\epsilon_0}\left\{\frac{1}{2}\ln\left[\frac{(x+\frac{L}{2})^2+y^2}{(x-\frac{L}{2})^2+y^2}\right]\hat{x} + \left(\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right)\hat{y}\right\}}$$

#4 continued

(b) Given the extremely complicated nature of the solution it is imperative to take the limit in various simple regime where we know the solution

(i) $x \to 0$: We must have $E_x \to 0$ since the charge distribution is symmetric about the y-axis

$$\lim_{x\to 0} \vec{E}(\vec{r}) = \frac{\sigma}{2\pi\epsilon_0}\left[\frac{1}{2}\ln\left[\frac{(L/2)^2+y^2}{(-L/2)^2+y^2}\right]\hat{x} + \left(\tan^{-1}\left(\frac{L}{2y}\right) - \tan\left(-\frac{L}{2y}\right)\right)\hat{y}\right]$$

$$= \frac{\sigma}{2\pi\epsilon_0}\left(2\tan^{-1}\left(\frac{L}{2y}\right)\right)\hat{y} \qquad \text{since } \ln(1) = 0$$

good!

(ii) $L \to \infty$: We must recover the expression for an infinite plane

$$\lim_{L\to\infty} \vec{E}(\vec{r}) = \frac{\sigma}{2\pi\epsilon_0}\lim_{L\to\infty}\left[\frac{1}{2}\ln\left[\frac{(L/2)^2}{(-L/2)^2}\right]\hat{x} + \left(\tan^{-1}\left(\frac{L}{2y}\right) - \tan^{-1}\left(-\frac{L}{2y}\right)\right)\hat{y}\right]$$

$$= \frac{\sigma}{2\pi\epsilon_0}\left(\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right)\hat{y} \qquad (y > 0)$$

$$= \frac{\sigma}{2\pi\epsilon_0}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)\hat{y} = \frac{\sigma}{2\epsilon_0}\hat{y} \qquad \text{yes!}$$

(Note: limit $L\to\infty$ is equivalent to $\frac{x}{L}\to 0$, $\frac{y}{L}\to 0$. Thus, very near the origin the strip looks like an infinite plane which makes sense physically)

(iii) $r \to \infty$: Far away from the axis, the strip will look like a line charge with $\lambda = \sigma L$ charge per unit length

$$\lim_{\substack{x\to 0\\ y\to\infty}} \vec{E}(\vec{r}) = \lim_{\substack{x\to 0\\ \frac{L}{y}\to 0}} \vec{E}(\vec{r})$$

Problem ~~4~~ continued

Using the result from b(i) $\lim_{x\to 0} \vec{E}(\vec{r}) = \frac{\sigma}{2\pi\epsilon_0}\left(2\tan^{-1}\left(\frac{L}{2y}\right)\right)\hat{y}$

Now $\lim_{\delta\to 0} \tan^{-1}(\delta) = \frac{d}{d\delta}\Big|_{\delta=0}(\tan^{-1}(\delta))\,\delta = \delta$

$\therefore \lim_{\frac{L}{y}\to 0} \tan^{-1}\left(\frac{L}{2y}\right) = \frac{L}{2y}$

$\therefore \lim_{\substack{x\to 0\\ y\to\infty}} \vec{E}(\vec{r}) = \frac{\sigma L}{2\pi\epsilon_0\, y}\hat{y} = \frac{1}{4\pi\epsilon_0}\left(\frac{2\lambda}{y}\right)\hat{y}$

where $\lambda = \sigma L$

This is just the field of an infinite line charge. When we get infinitely far away we can't tell that the charge is on a strip or a line. ("Infinitely far" means $r \gg L$)

Extra Credit

Now take the limit $L\to 0$ $\sigma\to\infty$. This is equivalent to

$$\lim_{\frac{L}{x}\to 0,\ \frac{L}{y}\to 0} \vec{E}(\vec{r}) = \frac{\sigma}{2\pi\epsilon_0}\left\{\frac{1}{2}\lim \ln\left[\frac{x^2+y^2+xL+L^2/4}{x^2+y^2-xL+L^2/4}\right]\hat{x} + \lim\left(\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right)\hat{y}\right\}$$

$$\lim_{L\to 0} \ln\left[\cdots\right] = \lim_{L\to 0} \ln\left[\frac{1+\frac{x}{x^2+y^2}L}{1-\frac{x}{x^2+y^2}L}\right] = \lim_{L\to 0} \ln\left[1+\frac{2x}{x^2+y^2}L\right]$$

$$= \left(\frac{2x}{x^2+y^2}\right)L \qquad \left(\text{Using } \lim_{\delta\to 0}\ln(1+\delta) = \delta\right)$$

Similarly

$$\lim_{L\to 0} \tan^{-1}\left(\frac{x+L/2}{y}\right) = \tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\frac{Ly}{x^2+y^2} \quad \left(\text{From Taylor expansion}\right)$$

$$\Rightarrow \lim_{L\to 0}\left[\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right] = \frac{yL}{x^2+y^2}$$

Putting this together

$$\lim_{\substack{L\to 0\\ \sigma\to\infty}} \vec{E}(r) = \frac{\sigma}{2\pi\epsilon_0}\left(\frac{xL}{x^2+y^2}\hat{x} + \frac{yL}{x^2+y^2}\hat{y}\right)$$

$$= \frac{\sigma L}{2\pi\epsilon_0}\left(\frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y}\right)\frac{1}{\sqrt{x^2+y^2}}$$

$$\boxed{\lim_{\substack{L\to 0\\ \sigma\to\infty}} = \frac{1}{4\pi\epsilon_0}\left(\frac{2\lambda}{\rho}\right)\hat{\rho} \quad \text{where } \lambda = \sigma L}$$

where $\rho = \sqrt{x^2+y^2}$

$$\hat{\rho} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} = \frac{x}{\rho}\hat{x} + \frac{y}{\rho}\hat{y}$$

This is of course what we expect:

the limit is equivalent to shrinking the strip and concentrating all the charge on a line

Problem 4: Part (c)

In[1]:=

```
Needs["Graphics`PlotField`"]
```

Define E-field: Note - I have added a condition for y=0 as the limit of the expression y-->0 so that the field doesn't blow up artificially. I didn't tell you do this this because I didn't want to add confusion.

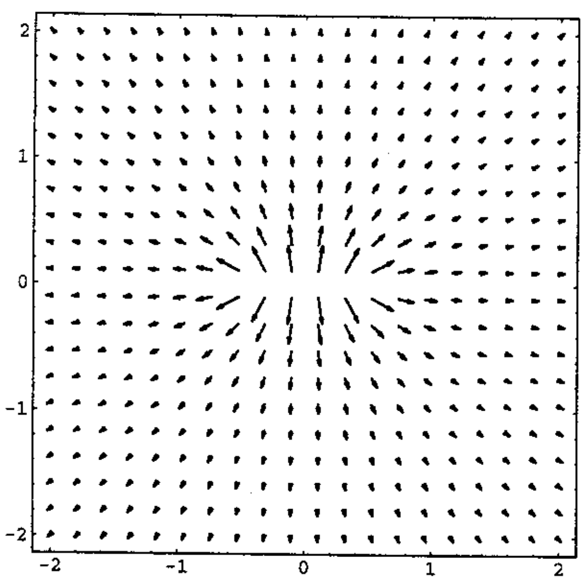
```
Efield[x_,y_,L_] :=
  If[y==0,
       { .5*Log[(x+L/2)^2/(x-L/2)^2] ,Pi},  (*the limt y->0*)
    (* Else, the genral case)
       {.5 Log[ ((x+L/2)^2+y^2) / ((x-L/2)^2+y^2) ] ,
       (ArcTan[(x+L/2)/y]-ArcTan[(x-L/2)/y])}      ]
```

Plots: Fixed range: -2<x<2, -2<y<2

(Note: I have added to fancy options to make the plots look better)

L=1 - Note the symmetry of the field lines

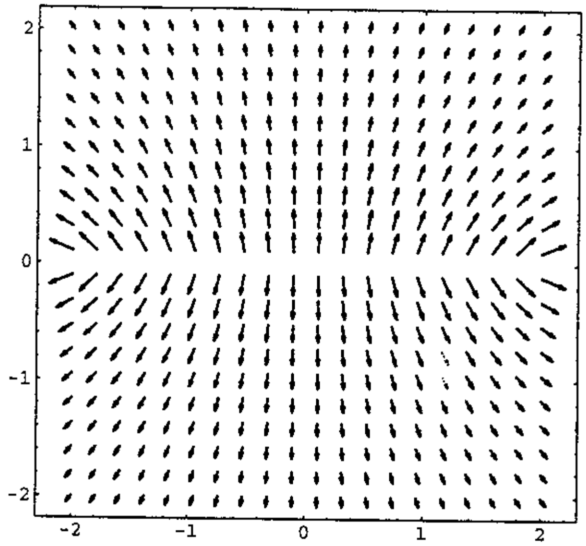
```
PlotVectorField[Efield[x,y,1],{x,-2,2},{y,-2,2},
                PlotPoints->20,Frame->True]
```



-Graphics-

■ **L=4 : For x and y on the order of L the field starts to look more like that of an infinite plane. We do see some "fringing" fields near the edge**

```
PlotVectorField[Efield[x,y,4],{x,-2,2},{y,-2,2},
        PlotPoints->20,Frame->True]
```



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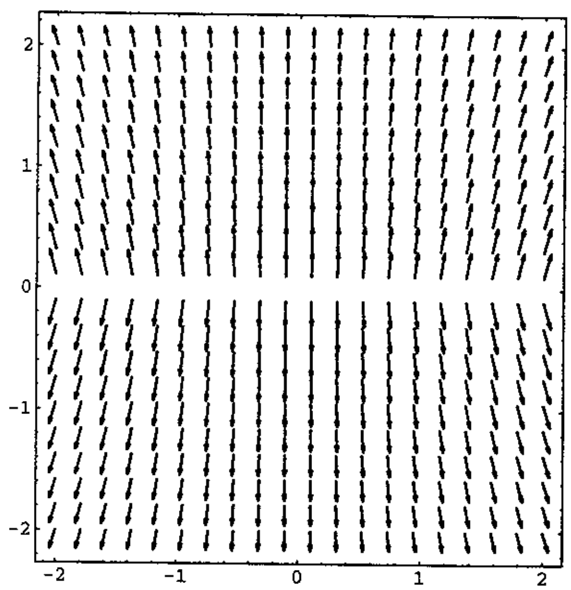
■ **L=10: For x and y small compared to L the field is very close to an infinite plane, (uniform in direction and length)**

```
PlotVectorField[Efield[x,y,10],{x,-2,2},{y,-2,2},
        PlotPoints->20,Frame->True]
```

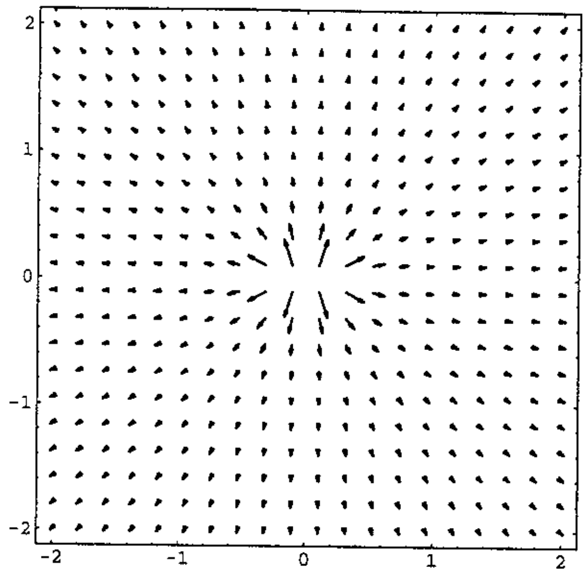


-Graphics-

■ **L=.5: For x and y large compared to L the field look like that of a line charge (line coming out of page)**

In[3]:=

```
PlotVectorField[Efield[x,y,.5],{x,-2,2},{y,-2,2},
                PlotPoints->20,Frame->True]
```



Out[3]=

```
-Graphics-
```

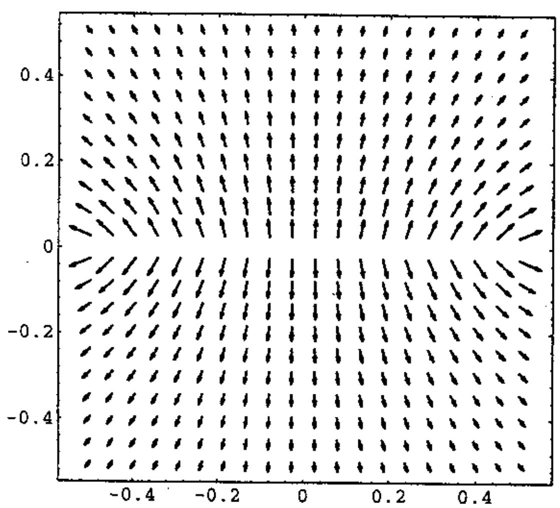
Plots: Fixed L=1:

The electric field depends only on the ***dimensionless ratios*** : x/L and y/L.

Thus we get identical plots to the one shown above by fixing L=1, and changing the plot region.

In[4]:=

```
PlotVectorField[Efield[x,y,1],{x,-1/2,1/2},{y,-1/2,1/2},
                PlotPoints->20,Frame->True]
```

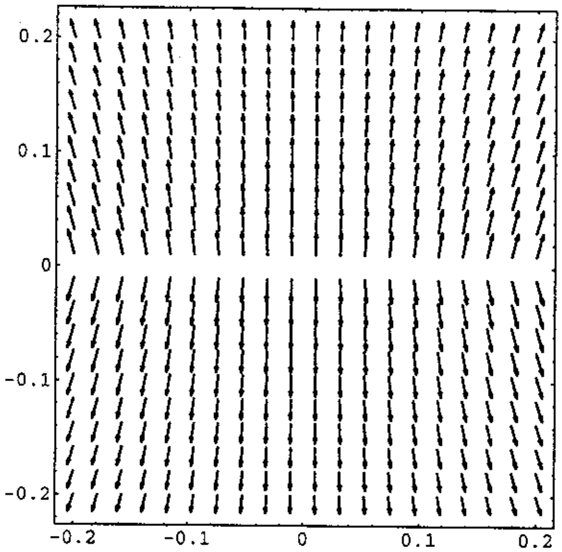


Out[4]=

```
-Graphics-
```

In[5]:=

```
PlotVectorField[Efield[x,y,1],{x,-1/5,1/5},{y,-1/5,1/5},
              PlotPoints->20,Frame->True]
```

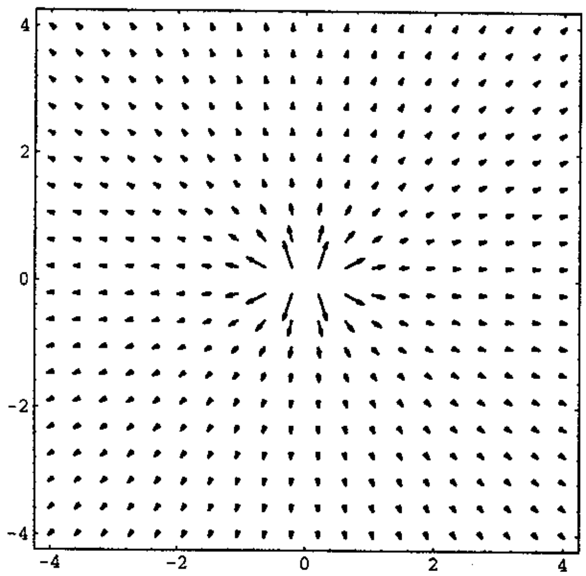


Out[5]=

-Graphics-

In[6]:=

```
PlotVectorField[Efield[x,y,1],{x,-4,4},{y,-4,4},
              PlotPoints->20,Frame->True]
```



Out[6]=

-Graphics-