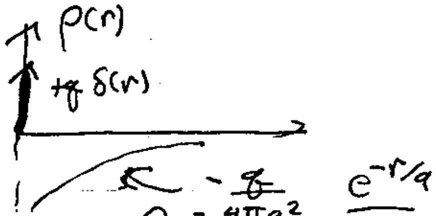
Physics 405

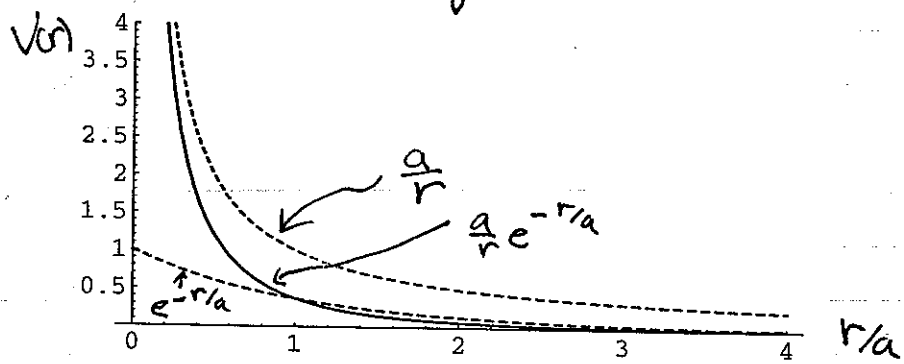
P.S. #4 Solutions

(1) The Yukawa Potential

Consider a charge distribution consisting of a point charge at the origin q, and a "smeared out" charge $-q$ which "screens" the long range force of the point



(a) We want to show that this charge distribution yields the Yukawa potential $V(r) = G \frac{e^{-r/a}}{r/a}$



For short distances (i.e. $r \ll a$) $e^{-r/a} \approx 1$ $\Rightarrow$ $V(r) \propto \frac{1}{r}$

For long distance (i.e. $r \gg a$) the exponential dominates

Thus, the Yukawa potential looks like a Coulomb potential for $r \ll a$, but falls off very rapidly to zero for $r \gg a$.

(b) The total charge is neutral: Proof

$$Q_{total} = q - \int d^3r\, \rho_-(\vec{r}) \quad , \quad \rho_-(\vec{r}) = -\frac{q}{4\pi a^2} \frac{e^{-r/a}}{r}$$

(1b) continued

- $Q_{total} = q\left[1 - \int_0^\infty \frac{e^{-r/a}}{4\pi a^2 r} 4\pi r^2 dr\right]$

Here I used the fact that ρ is spherically symmetric, so the volume element is just a spherical shell $d^3r = 4\pi r^2 dr$

Now change variables: let $u = r/a$

$\Rightarrow Q_{total} = q\left[1 - \int_0^\infty u e^{-u} du\right]$

Using integration by parts: $\frac{d}{du}(ue^{-u}) = e^{-u} - ue^{-u}$

$$\Rightarrow \int ue^{-u} du = \int \left(-\tfrac{d}{du}(ue^{-u}) + e^{-u}\right) du$$
$$= -e^{-u} - ue^{-u} = -e^{-u}(1+u)$$

$$\therefore Q_{total} = q\left[1 + e^{-u}(1+u)\Big|_0^\infty\right]$$
$$= q\left[1 + 0 - 1\right] = 0 \checkmark$$

This makes sense physically since $V(r)$ falls off at far distances much faster than $1/r$ so there is <u>no monopole moment</u>. In fact this charge distribution creates a potential which goes to zero at infinity faster than any multipole moment for a confined set of charges.

(1c) Since this charge distribution is spherically symmetric, Gauss' Law calls out to us.

First, from pure symmetry arguments, we know

$$\vec{E}(\vec{r}) = E(r)\,\hat{r} \quad \left(\begin{array}{l}\text{i.e. the magnitude of } \vec{E} \text{ depends} \\ \text{only on } r, \text{ not } \theta + \phi \\ \text{and points in radial direction}\end{array}\right)$$

$\Rightarrow$ If we consider a sphere centered at the origin

$$\Phi_E \equiv \oint_S \vec{E} \cdot \hat{n}\, dA = 4\pi r^2 E(r) \quad \text{where } r \text{ is the radius}$$

Gauss' Law: $\Phi_E = \dfrac{Q_{enclosed}}{\epsilon_0} \Rightarrow E(r) = \dfrac{Q_{enclosed}}{4\pi\epsilon_0 r^2}$

All we need now is the charge enclosed in the sphere

$$Q_{enclosed} = \int_V \rho(\vec{r})\, d^3r = \int_0^r \rho(r')\, 4\pi r'^2 dr' \quad (r' = \text{dummy})$$

$$= q\left(1 - \int_0^r \frac{e^{-r'/a}}{4\pi a^2 r'} 4\pi r'^2 dr'\right)$$

$$= q\left(1 - \int_0^{r/a} u\, e^{-u}\, du\right) \quad \left(\begin{array}{c}\text{same change of} \\ \text{variables}\end{array}\right)$$

$$= q\left(1 + e^{-u}(1+u)\Big|_0^{r/a}\right)$$

$$= q\left(1 + e^{-r/a}\left(1 + \frac{r}{a}\right) - 1\right)$$

$$\Rightarrow Q_{enc} = q\, e^{-r/a}\left(1 + \frac{r}{a}\right)$$

$$\therefore \vec{E}(r) = \frac{q}{4\pi\epsilon_0 r^2} e^{-r/a}\left(1 + \frac{r}{a}\right)\hat{r}$$

(Next Page)

(Ic) Continued

Now, to find the potential, use $\vec{E} = -\vec{\nabla} V$

$$\Rightarrow V(\vec{r}) = -\int_{\vec{r}_{ground}}^{\vec{r}} \vec{E}(\vec{r}') \cdot d\vec{l}$$

where $\vec{r}_{ground}$ is the point where $V=0$ and the path of integration is arbitrary since $\vec{\nabla}\times\vec{E} = 0$

For a confined charge distribution, choose $\vec{r}_{ground} = \infty$, and since $\vec{E}$ is along the radial direction, choose $d\vec{l} = \hat{r}\,dr$

$$\Rightarrow V(\vec{r}) = -\int_{\infty}^{r} E(r')\,dr' = \int_{r}^{\infty} \frac{q}{4\pi\epsilon_0 r'^2} e^{-r'/a}\left(1+\frac{r'}{a}\right) dr'$$

$$= \left(\frac{q}{4\pi\epsilon_0}\right)\frac{1}{a}\int_{r/a}^{\infty} du\, e^{-u}\left(\frac{1}{u^2}+\frac{1}{u}\right) \qquad \left(\text{where } u = r/a\right)$$

The integrand is a perfect derivative:

$$\frac{d}{du}\left(\frac{e^{-u}}{u}\right) = -\frac{e^{-u}}{u} - \frac{e^{-u}}{u^2} = -e^{-u}\left(\frac{1}{u^2}+\frac{1}{u}\right)$$

$$\therefore V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{a}\left[-\frac{e^{-u}}{u}\right]_{r/a}^{\infty}$$

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/a}}{r}}$$

Yukawa form with $G = \frac{1}{4\pi\epsilon_0}\frac{q}{a}$

(1d) The potential energy stored in the charge distribution, $\rho_-(\vec{r})$.

$$U = \frac{\epsilon_0}{2} \int_{\text{all space}} d^3r \; |\vec{E}_-(\vec{r})|^2$$

where here $\vec{E}_-(\vec{r})$ is the electric field arising from the negatively charged cloud only. We must, thus, substract off the field of the point charge. From part (c), we find

$$\vec{E}_-(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\left(e^{-r/a}\left(1+\frac{r}{a}\right) - 1\right)\hat{r}$$

$$\Rightarrow U = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_0^\infty \underbrace{(4\pi r^2 dr)}_{\text{Volume element}} \frac{\left(e^{-r/a}\left(1+\frac{r}{a}\right)-1\right)^2}{r^4}$$

Collecting constants, and letting $u = r/a$

$$\Rightarrow U = \frac{q^2}{8\pi\epsilon_0 a}\int_0^\infty du \left[e^{-2u}\left(\frac{1+u}{u}\right) - 2e^{-u}\left(\frac{1+u}{u^2}\right) + \frac{1}{u^2}\right]$$

Using integration by parts (or Mathematica)

$$U = \frac{q^2}{8\pi\epsilon_0 a}\left[-e^{-2u}\left(\frac{1}{u}+\frac{1}{2}\right) + \frac{2e^{-u}}{u} - \frac{1}{u}\right]_0^\infty$$

Must take limits at $\infty \xrightarrow{} U \Rightarrow 0$ and $0 \Rightarrow U \Rightarrow ?$

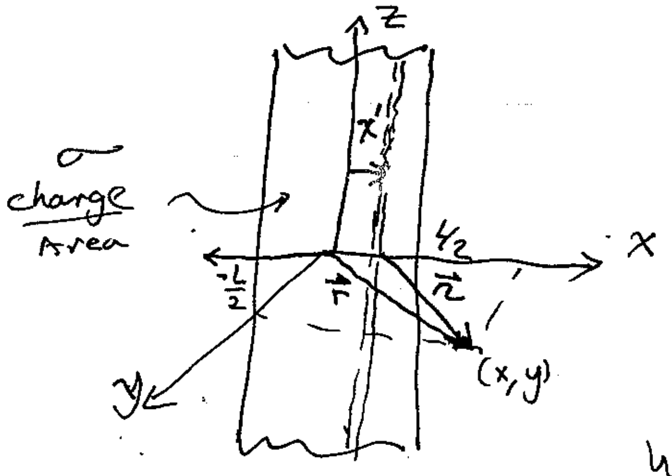
at 0

$$\Rightarrow U = \frac{-q^2}{8\pi\epsilon_0 a}\left[-\left(\frac{1}{u}+\frac{1}{2}\right) + \frac{2}{u} - \frac{1}{u}\right]_0 = \boxed{\frac{q^2}{16\pi\epsilon_0 a}}$$

Problem 2

a) Potential of a strip of charge

Back to the old geometry



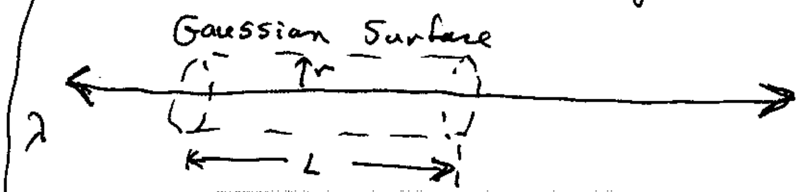
Break up the strip into substrip of thickness dx': These are infinite line charges, with charge per unit length

$$\lambda = \sigma dx'$$

We then integrate the contribution of each line charge as x' goes from $-\frac{L}{2} \to L/2$

Aside

What is the potential of an infinite line charge? This is most easily obtained from Gauss' Law:



$$\oint \vec{E} \cdot d\vec{a} = E(r)\, 2\pi r L$$

$$= \lambda L / \epsilon_0$$

$$\Rightarrow E(r) = \frac{\lambda}{2\pi r \epsilon_0}$$

Now this is tricky. The potential is defined by

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{l}$$, where $\vec{r}_0$ is ground

For an infinite line charge, we cannot take $\vec{r}_0$ at ∞. However, the choice of $\vec{r}_0$ is arbitrary. Choose a radial path $dl = \hat{r}\, dr$

$$V(r) = -\int_{r_0}^{r} E(r) dr = \left[-\ln(r) + \ln(r_0)\right] \frac{\lambda}{2\pi \epsilon_0}$$

For convenience, choose $r_0 = 1$ (arbitrary) $\quad V(r) = -\ln(r)$

OK, so now we have the potential of a line charge

$$V(r) = -\frac{\lambda}{2\pi\epsilon_0}\ln(r)$$ where r is the radial distance from the line to the point of observation.

Back to the problem at hand. The line charge at $\vec{r}' = x'\hat{x}$ contribute a potential dV at the observation point ~~~~ $\vec{r} = x\hat{x} + y\hat{y}$ of

$$dV(x,y;x') = -\frac{\lambda}{2\pi\epsilon_0}\ln(\mathscr{r}) \quad , \quad \lambda = \sigma\, dx'$$

where

$$\mathscr{r} = |\vec{\mathscr{r}}| = |\vec{r} - \vec{r}'| = |(x-x')\hat{x} + y\hat{y}|$$
$$= \sqrt{(x-x')^2 + y^2}$$

$$\Rightarrow dV(x,y;x') = -\frac{\sigma\, dx'}{2\pi\epsilon_0}\ln\left[\sqrt{(x-x')^2+y^2}\right]$$

Now integrate:

$$V(x,y) = \int_{x'=-L/2}^{x'=L/2} dV(x,y;x') = -\frac{\sigma}{2\pi\epsilon_0}\int_{-L/2}^{L/2} dx' \ln\left[\sqrt{(x-x')^2+y^2}\right]$$

let $u = x' - x$ $\qquad du = dx'$
$x' = u + x$

$$\Rightarrow V(x,y) = -\frac{\sigma}{2\pi\epsilon_0}\int_{x-L/2}^{x+L/2} du \ln\left[\sqrt{u^2+y^2}\right] =$$

$$-\frac{\sigma}{4\pi\epsilon_0}\int_{x-L/2}^{x+L/2} du \ln(u^2+y^2)$$

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From the integral given in the assignment

$$\int du \, \ln[u^2+y^2] = -2u + 2y\tan^{-1}\left(\frac{u}{y}\right) + u\ln[u^2+y^2]$$

$$\Rightarrow V(x,y) = -\frac{\sigma}{2\pi\epsilon_0}\left[-u + y\tan^{-1}\left(\frac{u}{y}\right) + \frac{u}{2}\ln[u^2+y^2]\right]_{x-L/2}^{x+L/2}$$

$$\boxed{\begin{aligned} V(x,y) = -\frac{\sigma}{4\pi\epsilon_0}\Big[-2L &+ 2y\left[\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right] \\ &+ (x+L/2)\ln\left[(x+L/2)^2+y^2\right] \\ &- (x-L/2)\ln\left[(x-L/2)^2+y^2\right]\Big] \end{aligned}}$$

(b) The electric field

$$\vec{E} = -\vec{\nabla}V = -\hat{x}\frac{\partial V}{\partial x} - \hat{y}\frac{\partial V}{\partial y} = E_x\hat{x} + E_y\hat{y}$$

$$\begin{aligned} -\frac{\partial V}{\partial x} = \frac{\sigma}{4\pi\epsilon_0}\Big\{ 2y&\left[\frac{1}{y}\left(\frac{1}{1+\left(\frac{x+L/2}{y}\right)^2}\right) - \frac{1}{y}\left(\frac{1}{1+\left(\frac{x-L/2}{y}\right)^2}\right)\right] \\ &+ \frac{2(x+L/2)^2}{y^2+(x+L/2)^2} - \frac{2(x-L/2)^2}{y^2+(x-L/2)^2} \\ &+ \ln\left[(x+L/2)^2+y^2\right] - \ln\left[(x-L/2)^2+y^2\right]\Big\} \end{aligned}$$

$$\begin{aligned} = \frac{\sigma}{4\pi\epsilon_0}\Big\{&\frac{2y^2+2(x+L/2)^2}{y^2+(x+L/2)^2} - \frac{2y^2+2(x-L/2)^2}{y^2+(x-L/2)^2} \\ &+ \ln\left[(x+L/2)^2+y^2\right] - \ln\left[(x-L/2)^2+y^2\right]\Big\} \end{aligned}$$

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$$\therefore \quad E_x = -\frac{\partial V}{\partial x} = \frac{\sigma}{4\pi\epsilon_0}\left\{ \ln\left[(x+L/2)^2+y^2\right] - \ln\left[(x-L/2)^2+y^2\right]\right\}$$

$$\boxed{E_x = \frac{\sigma}{2\pi\epsilon_0}\left\{\frac{1}{2}\ln\left[\frac{(x+L/2)^2+y^2}{(x-L/2)^2+y^2}\right]\right\}}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{\sigma}{4\pi\epsilon_0}\left\{2\left(\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right)\right.$$

$$+\ 2y\left(-\frac{(x+L/2)}{y^2}\,\frac{1}{1+\left(\frac{x+L/2}{y}\right)^2} + \frac{(x-L/2)}{y^2}\,\frac{1}{1+\left(\frac{x-L/2}{y}\right)^2}\right)$$

$$\left. +\ (x+L/2)\,\frac{2y}{y^2+(x+L/2)^2} - (x-L/2)\,\frac{2y}{y^2+(x-L/2)^2}\right\}$$

$$\Rightarrow E_y = \frac{\sigma}{2\pi\epsilon_0}\left\{\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right.$$

$$-\frac{y(x+L/2)}{y^2+(x+L/2)^2} + \frac{y(x+L/2)}{y^2+(x+L/2)^2}$$

$$\left. +\ \frac{y(x-L/2)}{y^2+(x-L/2)^2} - \frac{y(x-L/2)}{y^2+(x-L/2)^2}\right\}$$

$$\Rightarrow \boxed{E_y = \frac{\sigma}{2\pi\epsilon_0}\left\{\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right\}}$$

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(G) limit $L \to \infty$ (i.e $x << L$, $y << L$)

$$\lim_{L\to\infty}\left[\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right] = \pi \quad \text{(From P.S. \#3)}$$

Now things get tricky again

$$\text{limit}_{L\to\infty}\left\{(x+L/2)\ln\left[(x+L/2)^2+y^2\right] - (x-L/2)\ln\left[(x-L/2)^2+y^2\right]\right)$$

$$= \lim_{L\to\infty}\left(L\ln(L/2) + L\ln(L/2)\right)$$

$$= \lim_{L\to\infty}\left[2L\ln(L/2)\right]$$

So $$\lim_{L\to\infty} V(x,y) = -\frac{\sigma}{2\epsilon_0}y + \underbrace{\lim_{L\to\infty}\left(-2L + 2L\ln(L/2)\right)}_{?}$$

The term in brackets blows up, but thats really an artifact of the problem of where to put ground. In general we can add or subtract any constant from V with out effecting the physics

So $$\boxed{\lim_{L\to\infty} V(x,y) = \frac{\sigma}{2\epsilon_0}y + \text{constant}}$$

Good: This is the potential of an infinite plane

$$\vec{E} = -\vec{\nabla}V = \frac{\sigma}{2\epsilon_0}\hat{y} \quad (y>0)$$ ✓

(E) continued

Now take the limit as $x, y \to \infty$

- $$\lim_{x,y\to\infty}\left[\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right]$$

$$= \lim_{x,y\to\infty}\left[\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x}{y}\right)\right] = 0$$

- $$\lim_{x,y\to\infty}\left\{(x+L/2)\ln\left[(x+\tfrac{L}{2})^2+y^2\right] - (x-L/2)\ln\left[(x-L/2)^2+y^2\right]\right\}$$

$$= L\ln(x^2+y^2) = L\ln(r^2) = 2L\ln(r)$$

$$\therefore \lim_{x,y\to\infty} V(x,y) = -\frac{\sigma}{4\pi\epsilon_0}\left(-2L + 2L\ln(r)\right)$$

$$\boxed{\lim_{x,y\to\infty} V(x,y) = -\frac{\sigma L}{2\pi\epsilon_0}\ln(r) + \text{Const}}$$

Good, this is the potential of an infinite line charge with $\lambda = \sigma L$ the $\frac{\text{charge}}{\text{length}}$

```
Needs["Graphics`PlotField`"]
```

In[21]:=

```
V[x_,y_] := 2y (ArcTan[(L/2-x)/y] + ArcTan[(L/2+x)/y]) -
              (x-L/2) Log[(x-L/2)^2+y^2] + (x+L/2) Log[(x+L/2)^2+y^2]
```

In[22]:=

```
L=1  (*  Choosing L=1 to set the scale  *)
```

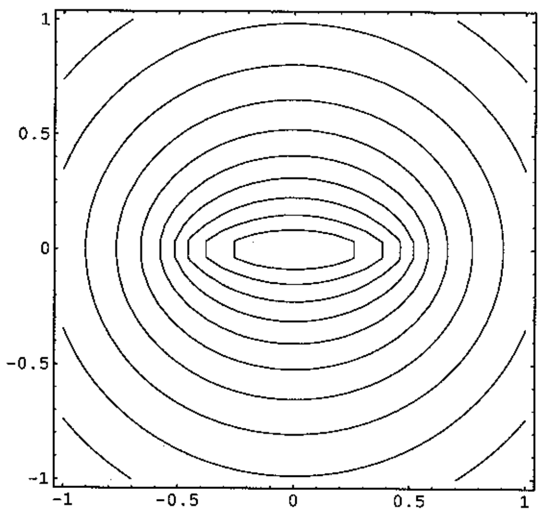
Out[22]=

```
1
```

Single Positively charged strip

x and y on the order of L

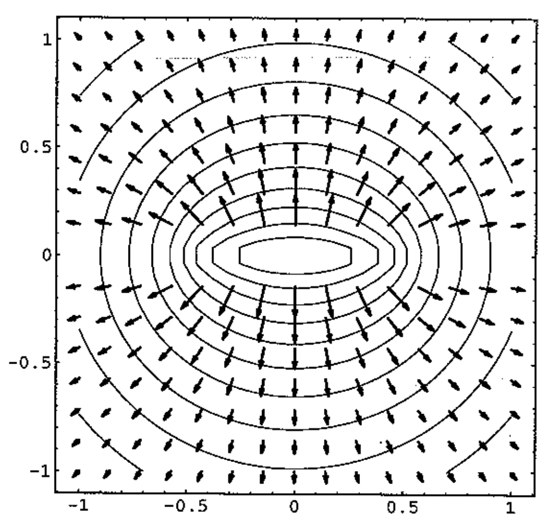
```
ContourPlot[V[x,y], {x,-1,1}, {y,-1,1}, ContourShading->False,
            PlotPoints->30]
```



-ContourGraphics-

```
PlotGradientField[V[x,y], {x,-1,1}, {y,-1,1}]
```

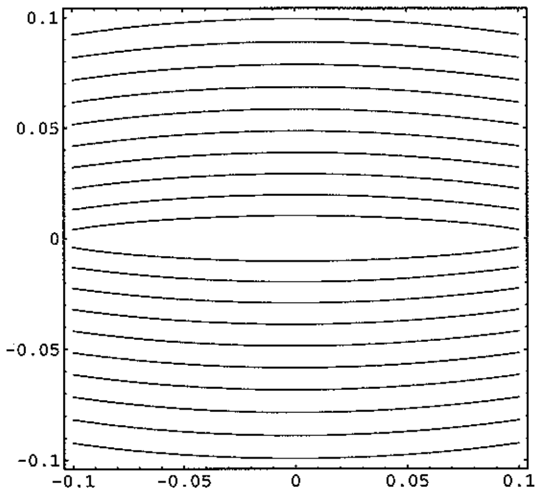
```
Show[%24,%25]
```



-Graphics-

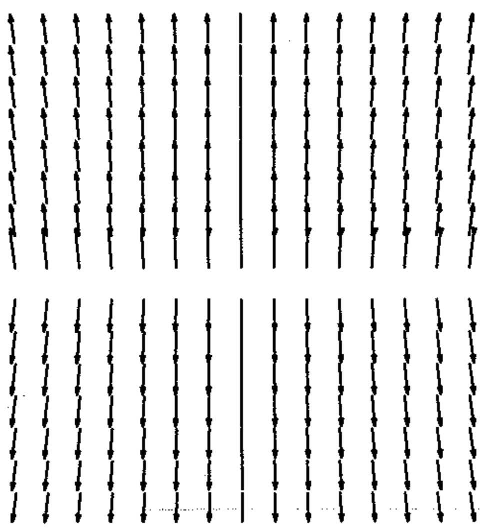
x and y on the small compared to L   (looks more like a plane)

```
ContourPlot[V[x,y], {x,-.1,.1}, {y,-.1,.1}, ContourShading->False,
            PlotPoints->30]
```



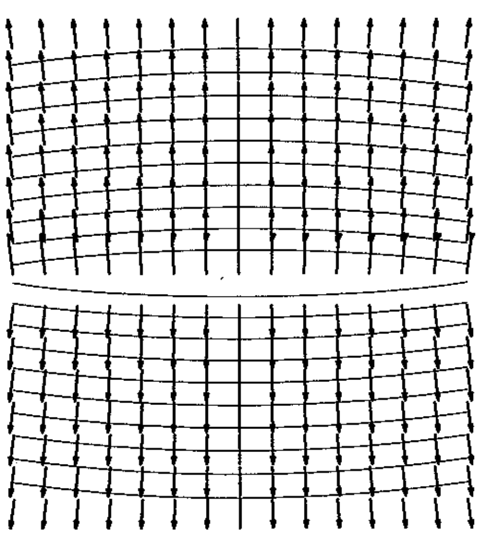
-ContourGraphics-

```
PlotGradientField[V[x,y], {x,-.1,.1}, {y,-.1,.1}]
```



-Graphics-

```
Show[%,%%]
```

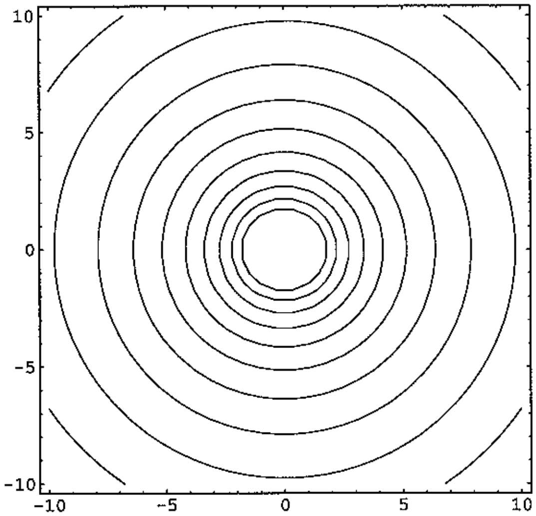


-Graphics-

■ ~~x and y on the big compared to L (looks more like a line charge)~~

In[24]:=

```
ContourPlot[V[x,y], {x,-10,10}, {y,-10,10}, ContourShading->False,
        PlotPoints->30]
```

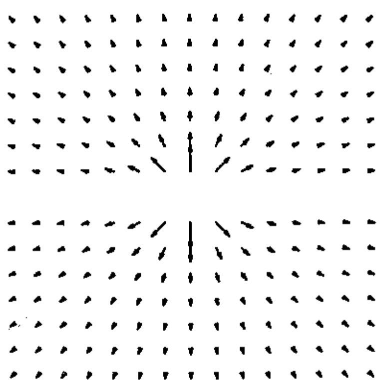


Out[24]=

```
-ContourGraphics-
```

In[26]:=

```
PlotGradientField[V[x,y], {x,-10,10}, {y,-10,10}]
```

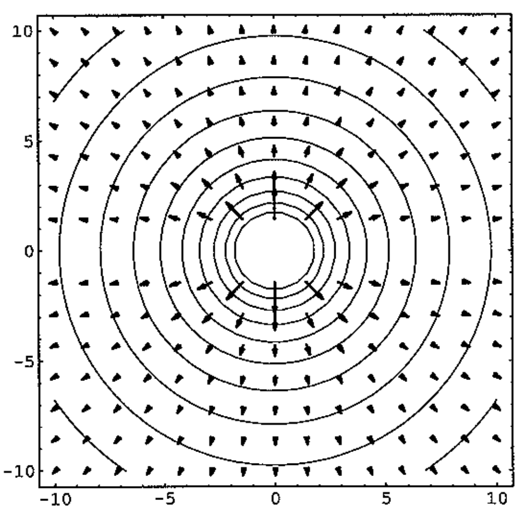


Out[26]=

```
-Graphics-
```

In[27]:=

```
Show[%24,%26]
```



Out[27]=

```
-Graphics-
```

Two oppositely charged strips

In[28]:=

Principle of superposition!

```
V2[x_,y_] = V[x,y] - V[x,y+s]
```

Out[28]=

$$2\,y\,\left(\mathrm{ArcTan}\left[\frac{\frac{1}{2}-x}{y}\right]+\mathrm{ArcTan}\left[\frac{\frac{1}{2}+x}{y}\right]\right)-2\,(s+y)\,\left(\mathrm{ArcTan}\left[\frac{\frac{1}{2}-x}{s+y}\right]+\mathrm{ArcTan}\left[\frac{\frac{1}{2}+x}{s+y}\right]\right)-$$
$$\left(-\left(\tfrac{1}{2}\right)+x\right)\,\mathrm{Log}\left[\left(-\left(\tfrac{1}{2}\right)+x\right)^2+y^2\right]+\left(\tfrac{1}{2}+x\right)\,\mathrm{Log}\left[\left(\tfrac{1}{2}+x\right)^2+y^2\right]+$$
$$\left(-\left(\tfrac{1}{2}\right)+x\right)\,\mathrm{Log}\left[\left(-\left(\tfrac{1}{2}\right)+x\right)^2+(s+y)^2\right]-\left(\tfrac{1}{2}+x\right)\,\mathrm{Log}\left[\left(\tfrac{1}{2}+x\right)^2+(s+y)^2\right]$$

spacing s on the order of the thickness L

In[30]:=

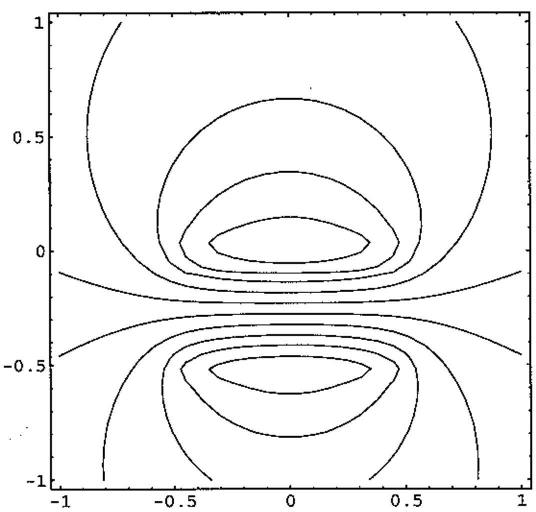
```
s=.5
```

Out[30]=

```
0.5
```

In[31]:=

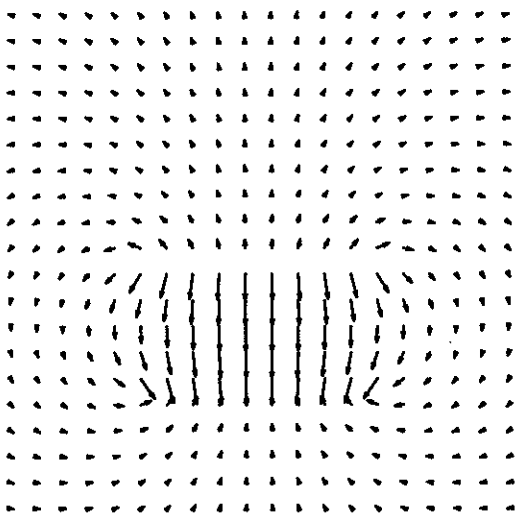
```
ContourPlot[V2[x,y], {x,-1,1}, {y,-1,1}, ContourShading->False,
            PlotPoints->30]
```



Out[31]=

```
-ContourGraphics-
```

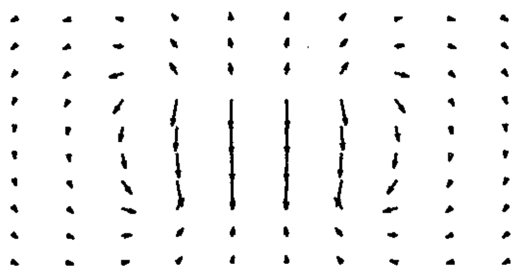
```
PlotGradientField[V2[x,y], {x,-1,1}, {y,-1,1},
                  PlotPoints->20]
```



Out[32]=

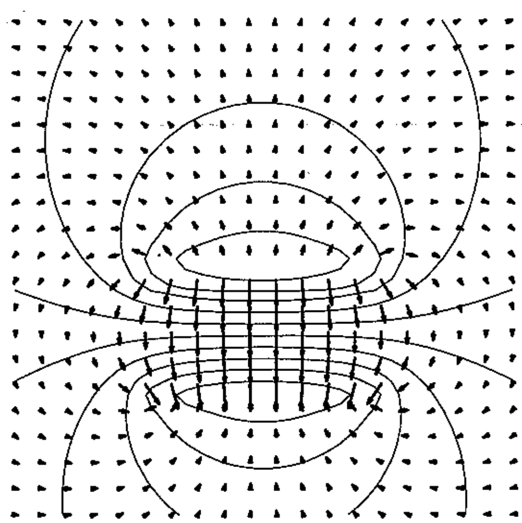
-Graphics-

```
PlotGradientField[V2[x,y], {x,-1,1}, {y,-.75,.25},
                  PlotPoints->10]
```



-Graphics-

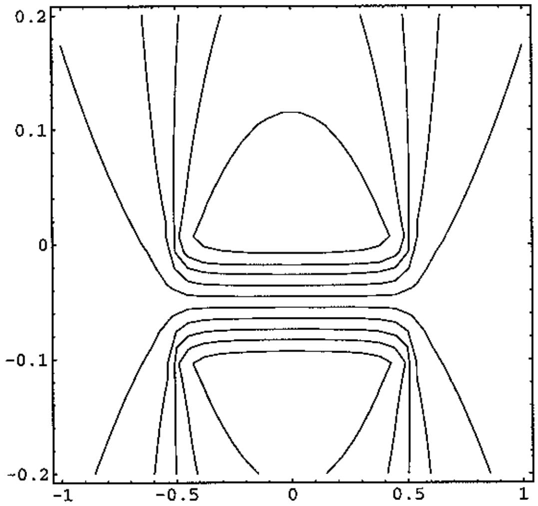
```
Show[%,%%]
(* Field somewhat uniform inside.  Smaller fringing fields outside*)
```



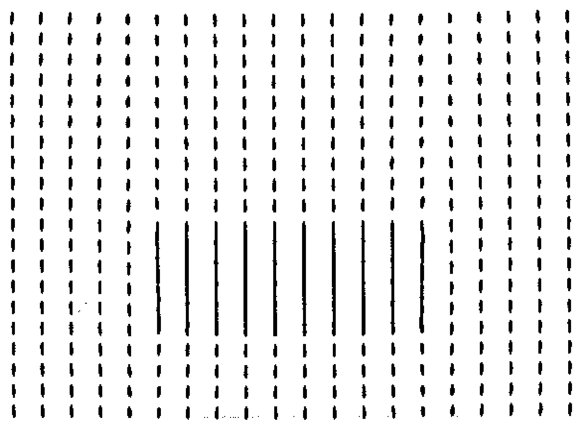
-Graphics-

~~Spacing s small compared the width L~~

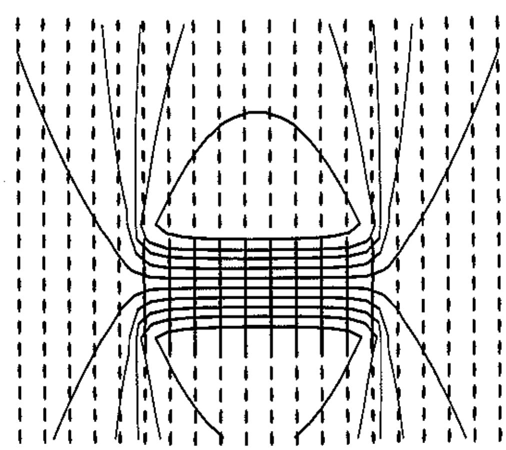
```
s=.1

0.1

ContourPlot[V2[x,y],{x,-1,1}, {y,-.2,.2}, ContourShading->False,
            PlotPoints->30]
```




```
-ContourGraphics-

PlotGradientField[V2[x,y], {x,-1,1}, {y,-.2,.2},
                  PlotPoints->20]
```



```
-Graphics-

Show[%19,%20]
(* Field very uniform inside, very little fringing fields *)
```

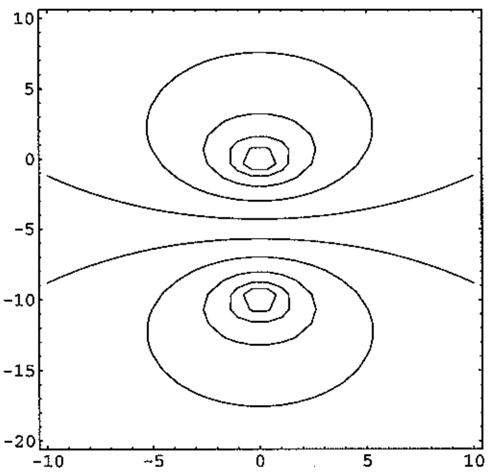


```
-Graphics-
```

Spacing s very large compared to width L

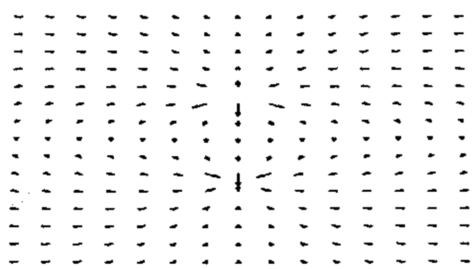
In[33]:=
```
s=10
```
Out[33]=
```
10
```

In[34]:=
```
ContourPlot[V2[x,y],{x,-10,10}, {y,-20,10}, ContourShading->False,
            PlotPoints->30]
```



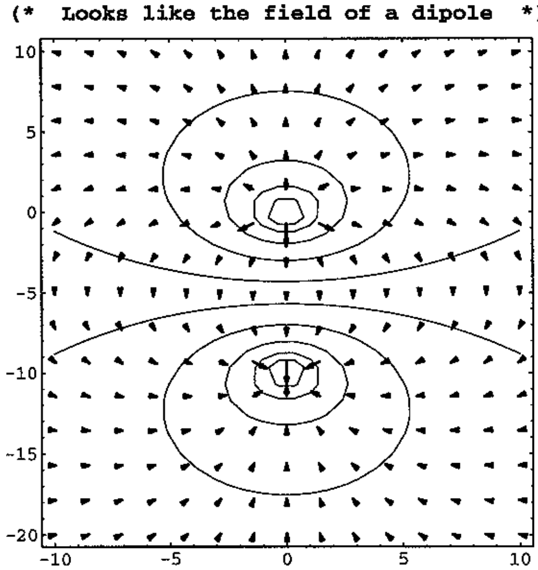
Out[34]=
```
-ContourGraphics-
```

In[36]:=
```
PlotGradientField[V2[x,y],{x,-10,10}, {y,-20,10}]
```



Out[36]=
```
-Graphics-
```

```
Show[%34,%36]

(*  Looks like the field of a dipole   *)
```



Out[37]=
```
-Graphics-
```