(1) Griffiths, Problem 2.35

Charge $q_e$ on the surface of a conducting sphere of radius $R$ surrounded by a conducting spherical shell, inner radius $a$ and outer radius $b$.

(a) The field inside the conductor is zero. Therefore, by Gauss' Law:

$$\frac{4\pi r^2 E(r)}{\varepsilon_0} = \frac{1}{\varepsilon_0} Q_{\text{enc}} = \frac{1}{\varepsilon_0} \left( q_e + \sigma - 4\pi \right)$$

Flux of electric field (due to spherical sym.) = 0 (conductor)

$$\Rightarrow \sigma a = -\frac{q_e}{4\pi a^2}$$

Because the conductor is overall neutral,

$$4\pi a^2 \sigma_a = -4\pi b^2 \sigma_b$$

Total charge at $r = a$: $-\sigma_a a$, $r = b$: $\sigma_b b$.

$$\Rightarrow 0 = \sigma_b - \frac{q_e}{4\pi b^2}$$
(b) \( V(0) = - \int_{\infty}^{0} E(r) \, dr \) (spherical symmetry, \( r = \infty \) at source)

Using Gauss' Law, for spherical symmetry:

\[
E(r) = \frac{Q_{enc}(r)}{4\pi \epsilon_0 \ r^2}
\]

\[
Q_{enc}(r) = \begin{cases} \frac{q}{2} & R < r < a \\ \frac{q}{2} & a < r < b \\ 0 & r > b \end{cases}
\]

\( \Rightarrow E(r) = \begin{cases} 0 & r < R \\ \frac{q}{4\pi \epsilon_0 \ r^2} & R < r < a \\ 0 & a < r < b \\ \frac{q}{4\pi \epsilon_0 \ r^2} & r > b \end{cases} \)

(\text{Check that this satisfies the boundary conditions!})

\( \Rightarrow V(0) = \int_{R}^{b} \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} \, dr + \int_{a}^{b} \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} \, dr \)

\[
V(0) = \frac{q}{4\pi \epsilon_0} \left( \frac{1}{R} - \frac{1}{a} + \frac{1}{b} \right)
\]

A general graph of the potential as a function of \( r \) is:

Notice that \( V(r) \) has discontinuity in its derivative at the surfaces of the conductor. (Does this make sense?)
(c) The outer surface is grounded $\Rightarrow V(b) = V(c) = 0$

\[ V_a = -\frac{q}{4\pi \epsilon_0 a^2} \]  (required to keep $E = 0$ inside shell)

**No change**

\[ \sigma_{r0} = 0 \]  (required to keep $V = 0 = V(c)$)

**This changes!**

Now $V(0) = \frac{-\int_0^0 \overline{E}(r) \, dr}{a} = -\int_0^0 \overline{E}(r) \, dr$  (Since $V = 0$ for $r > a$).

\[ = -\int_a^R \overline{E}(r) \, dr = -\frac{q}{4\pi \epsilon_0} \int_a^R \frac{dr}{r^2} = \frac{q}{4\pi \epsilon_0} \left( \frac{1}{R} - \frac{1}{a} \right) \]

**This changes!**

New graph of potential: $V_{\text{new}} = \begin{cases} V_{\text{old}} - \frac{q}{4\pi \epsilon_0} \frac{1}{r} & r \leq a \\ 0 & r > a \end{cases}$

\[ V = \frac{q}{4\pi \epsilon_0} \left( \frac{1}{r} - \frac{1}{a} \right) \]
(2) Griffiths, Problem 2.39

Find capacitance/length.

To find the capacitance we distribute charge \( +Q \) on the surface of one conductor and charge \( -Q \) on the surface of the other and find the potential difference:

\[ V_{+ -} = \int_{+}^{\text{edge}} E \cdot dl \]

For a long cylinder \( L \gg b \), we can ignore fringing fields and approximate the cylinder as infinite. By Gauss' Law:

\[ (2\pi r L) E(r) = \frac{Q_{enc} (r)}{\varepsilon_0} \Rightarrow E(r) = \frac{1}{2\pi \varepsilon_0} \left( \frac{Q_{enc}(r)}{r} \right) \]

\[ = \oint E \cdot dA \quad \text{(for cylindrical symmetry)} \]

here \( \lambda_{enc}(r) = \frac{Q_{enc}}{L} = \text{Charge per unit length enclose in the Gaussian cylinder, radius } r \)

For the problem at hand:

\[ \lambda_{enc}(r) = \begin{cases} 
0 & r < a \\
\frac{Q}{L} & a < r < b \\
0 & r > b 
\end{cases} \]

\[ \Rightarrow \quad E(r) = \begin{cases} 
0 & r < a \\
\frac{Q}{L \cdot 2\pi \varepsilon_0} \frac{1}{r} & a < r < b \\
0 & r > b 
\end{cases} \]
The potential difference \( V_{+} \) is then
\[
V_{+} = - \int_{a}^{b} E(r) \, dr = \frac{Q}{2 \pi \varepsilon_0 L} \left( \int_{a}^{b} \frac{dr}{r} \right)
\]
\[
= \frac{Q}{2 \pi \varepsilon_0 L} \ln \left( \frac{b}{a} \right)
\]

By definition, \( C = \frac{Q}{V_{+}} \) (the capacitance)

\[The \ capacitance/length\]
\[
\frac{C}{L} = \frac{Q/L}{V_{+}} = \frac{2 \pi \varepsilon_0}{\ln \left( \frac{b}{a} \right)}
\]

The total potential energy stored in the field can be calculated by
\[
U = \frac{\varepsilon_0}{2} \int_{d^3r} \frac{V^2}{(\nabla \cdot \mathbf{E})^2} \; d^3r = \frac{\varepsilon_0}{2} \int_{d^3r} \frac{V^2}{(\nabla \cdot \mathbf{E})^2} \; d^3r
\]
with \( d^3r = 2\pi r dr \, d\theta \, dz \) (cylindrical coordinates)

Since \( E = 0 \) except for \( a < r < b \)
\[
U = \frac{\varepsilon_0}{2} \left( \frac{Q}{2 \pi \varepsilon_0 L} \right)^2 2\pi L \int_{a}^{b} r \, dr \frac{1}{r^2}
\]
\[
= \frac{Q^2}{8 \pi \varepsilon_0 L} \ln \left( \frac{b}{a} \right) = \frac{1}{2} \left( \frac{2 \pi \varepsilon_0 L}{\ln \left( \frac{b}{a} \right)} \right) \left( \frac{Q}{2 \pi \varepsilon_0 L} \right) \frac{1}{V^2} \frac{1}{C}
\]
\[
= \frac{1}{2} CV^2 = \text{Work necessary to assemble charges on the conductor (read Griffiths)}
\]
In the limit that the spacing between the cylinders goes to zero, we expect the field between to become more and more uniform, and so the coaxial lines should look like a parallel plate.

Field almost uniform inside

\[ b = a + d \]
\[ \frac{d}{a} \ll 1 \]

\[ C/L = \frac{2\pi \varepsilon_0}{\ln(b/a)} = \frac{2\pi \varepsilon_0}{\ln(1 + \frac{d}{a})} \]

Now use the first order Taylor expansion:
\[ \ln(1 + S) \approx S \text{ for } S \ll 1 \]

\[ \Rightarrow \quad \frac{C}{L} \approx \frac{2\pi \varepsilon_0}{d/a} \Rightarrow C \approx \varepsilon_0 \frac{2\pi \varepsilon_0}{d} \]

Aside: \( 2\pi L = \text{Area of overlap} \equiv A \)

\[ C \approx \varepsilon_0 \frac{A}{d} \text{ as expected} \]