Physics 405

Problem Set #7 Solutions

Problem 1 (Problem 3.18)

A spherical shell has a fixed potential on the surface

$$V(R,\theta) = k\cos(3\theta), \quad \text{also } V \to 0 \text{ as } r \to \infty$$

Because the b.c.'s have no ϕ dependence, the general solution has the form

$$V(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Outside the sphere, $r \geq R$, $V \to 0$ as $r \to \infty \Rightarrow A_\ell = 0 \ \forall \ell$

$$V_{out}(r,\theta) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Use b.c. at $r = R$ to find $\{B_\ell\}$

$$V_{out}(R,\theta) = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = k\cos(3\theta)$$

Now we can find $\{B_\ell\}$ by integrating with the appropriate Legendre polynomial. However we can simplify the calculation by using appropriate trig identities. We want to express $\cos(3\theta)$ as a polynomial in $\cos\theta$

P.S. #7

Let us write

$$\cos(3\theta) = \cos(\theta + 2\theta) = \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$$

$$= \cos\theta\,(\cos^2\theta - \sin^2\theta) - \sin\theta\,(2\sin\theta\cos\theta) \qquad \left(\text{Double angle formula}\right)$$

$$= \cos\theta\,(2\cos^2\theta - 1) - 2(1 - \cos^2\theta)\cos\theta \qquad (\text{use } \cos^2\theta = 1 - \sin^2\theta)$$

$$= 4\cos^3\theta - 3\cos\theta$$

Let $x = \cos\theta$

$$\Rightarrow \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(x) = k(4x^3 - 3x)$$

Again, we may use $\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \begin{cases} \frac{2}{2\ell+1} & \ell = \ell' \\ 0 & \ell \neq \ell \end{cases}$

but for this simple polynomial we can easily express the right hand side as a small sum of P_ℓ's

Recal $P_0(x) = 1$, $P_2(x) = x$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, etc.

$$\therefore x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$$\therefore 4x^3 - 3x = \frac{8}{5}P_3(x) + \frac{12}{5}P_1(x) - 3P_1(x) = \frac{8}{5}P_3(x) - \frac{3}{5}P_1(x)$$

$$\therefore \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(x) = k\left(-\frac{3}{5}P_1(x) + \frac{8}{5}P_3(x)\right)$$

$$\Rightarrow \frac{B_1}{R^2} = -\frac{3}{5}k \Rightarrow B_1 = -\tfrac{3}{5}R^2 k$$

$$\frac{B_3}{R^4} = \frac{8}{5}k \Rightarrow B_3 = \frac{8}{5}R^4 k$$

$B_\ell = 0$ for all other ℓ

Check: Using "orthogonality" of P_ℓ's

$$\int_{-1}^{1} dx\, P_m(x) \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(x) = \int_{-1}^{1} dx\, P_m(x)\,(4x^3-3x)$$

$$\Rightarrow \quad B_m = R^{m+1}\,\frac{2m+1}{2} \int_{-1}^{1} dx\, P_m(x)\,(4x^3-3x)$$

Since $4x^3-3x$ is an odd function of x, the right hand side vanishes for m-even. In addition the polymial in the integrand is order-3 so $B_m=0$ for $m>3$

$$\therefore \quad B_1 = R^2\,\frac{3}{2}\int_{-1}^{1} dx\, P_1(x)\,(4x^3-3x) = R^2\,\frac{3}{2}\int_{-1}^{1} dx\, x\,(4x^3-3x)$$

$$= R^2\,\frac{3}{2}\left(\frac{4}{5}x^5 - x^3\right)\Big|_{-1}^{1} = R^2\, 3\left(\frac{4}{5}-1\right) = -\frac{3}{5}R^2 \checkmark$$

$$B_3 = R^4\,\frac{7}{2}\int_{-1}^{1} dx\, P_3(x)\,(4x^3-3x) = R^4\,\frac{7}{2}\int_{-1}^{1} dx \left(\frac{5x^3-3x}{2}\right)(4x^3-3$$

$$= R^4\,\frac{7}{4}\int_{-1}^{1} dx\,(20x^6 - 27x^4 + 9x^2)$$

$$= R^4\,\frac{7}{4}\left(\frac{20}{7}x^7 - \frac{27}{5}x^5 - 3x^3\right)\Big|_{-1}^{1}$$

$$= R^4\,\frac{7}{2}\left(\frac{20}{7} - \frac{27}{5} - 3\right) = \frac{8}{5}R^4 \checkmark$$

$\therefore$ <u>Outside</u> $r \geq R$

$$V_{out}(r,\theta) = \frac{B_1}{r^2}P_1(\cos\theta) + \frac{B_3}{r^4}P_3(\cos\theta)$$

$\underline{r \geq R}$ $$\boxed{V_{out}(r,\theta) = \frac{k}{5}\left(-3\left(\frac{R}{r}\right)^2 P_1(\cos\theta) + 8\left(\frac{R}{r}\right)^4 P_3(\cos\theta)\right)}$$

Inside the sphere $r \le R$

Must have V finite at $r=0$

$$\Rightarrow B_\ell = 0 \ \forall \ell \quad \Rightarrow \quad V_{in}(r,\theta) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$

To find $\{A_\ell\}$, insist on the continuity of the potential at the surface of the sphere ($r=R$

$$\Rightarrow V_{in}(R,\theta) = V_{out}(R,\theta)$$

$$\therefore \sum_\ell A_\ell R^\ell P_\ell(\cos\theta) = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow A_\ell = \frac{B_\ell}{R^{2\ell+1}} \Rightarrow \begin{cases} A_1 = \dfrac{B_1}{R^3} = -\dfrac{3}{5}k\dfrac{1}{R} \\ A_3 = \dfrac{B_3}{R^7} = -\dfrac{3}{5}k\dfrac{1}{R^3} \\ A_\ell = 0 \quad \text{otherwise} \end{cases}$$

$\therefore$ Inside $(r \le R)$

$$\boxed{V_{in}(r,\theta) = \frac{k}{5}\left(-3\left(\frac{r}{R}\right) P_1(\cos\theta) + 8\left(\frac{r}{R}\right)^3 P_3(\cos\theta)\right)}$$

Surface Charge density (Use boundary condition on $\vec{E}_\perp$)

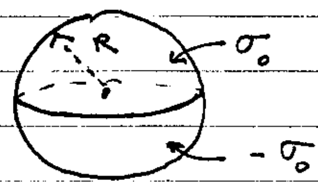
$$\Delta E_\perp = \frac{\sigma}{\epsilon_0} \Rightarrow \left(-\frac{\partial V_{out}}{\partial r}\bigg|_{r=R}\right) - \left(-\frac{\partial V_{in}}{\partial r}\bigg|_{r=R}\right) = \frac{\sigma(\theta)}{\epsilon_0}$$

$$\Rightarrow \frac{\sigma}{\epsilon_0}(\theta) = \frac{k}{5}\left\{-\frac{6}{R}P_1(\cos\theta) + \frac{32}{R}P_3(\cos\theta) - \frac{3}{R}P_1(\cos\theta) + \frac{24}{R}P_3(\cos\theta)\right.$$

$$\Rightarrow \frac{\sigma(\theta)}{\cancel{}} = \frac{k\epsilon_0}{5R}\left\{-9\cos\theta + 56\,P_3(\cos\theta)\right\}$$

Problem 2 Griffiths 3.22

A spherical shell, radius R, carries uniform surface charge σ_0 on the northern hemisphere and $-\sigma_0$ on southern h.s.



We can write the surface charge as a function of the polar angle θ

$$\sigma(\theta) = \begin{cases} \sigma_0 & 0 \le \theta \le \pi/2 \\ -\sigma_0 & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$

The solution follows as in Griffiths example 9 pp 142-144

- Find $V_{out}(r,\theta)$ $\quad r \ge R$

  $V_{in}(r,\theta)$ $\quad r \le R$

- Match solution at boundary

$$V_{out}(R,\theta) = V_{in}(R,\theta) \quad \text{(potential continuous)}$$

$$-\frac{\partial V_{out}(R,\theta)}{\partial r} + \frac{\partial V_{in}(R,\theta)}{\partial r} = \frac{\sigma(\theta)}{\epsilon_0} \quad \left(\begin{array}{l}\text{Normal component of} \\ \vec{E} \text{ discontinuous}\end{array}\right)$$

- Solution for $r \ge R$

  Potential must go to zero at infinity

$$\Rightarrow V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

- Solution for $r \le R$

  Potential must not blow up at $r=0$

$$\Rightarrow V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

· Matching the solution at the boundary

(i) Continuity of the potential.

$$V_{in}(R,\theta) = V_{out}(R,\theta)$$

$$\sum_{\ell} A_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow B_\ell = A_\ell R^{2\ell+1}$$

$$\therefore \quad V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \qquad r \leq R$$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \frac{R^{2\ell+1}}{r^{\ell+1}} P_\ell(\cos\theta) \qquad r \geq R$$

(ii) discontinuity of $E_\perp = E_r$ at $r = R$

$$E_r^{in} = \frac{-\partial V_{in}(r,\theta)}{\partial r} = -\sum_{\ell=0}^{\infty} \ell A_\ell r^{\ell-1} P_\ell(\cos\theta) \qquad r \leq R$$

$$E_r^{out} = -\frac{\partial V_{out}(r,\theta)}{\partial r} = +\sum_{\ell=0}^{\infty} (\ell+1) A_\ell \frac{R^{2\ell+1}}{r^{\ell-2}} P_\ell(\cos\theta) \qquad r \geq R$$

$$\Delta E_\perp\Big|_{r=R} = E_r^{out}(R,\theta) - E_r^{in}(R,\theta) =$$

$$= \sum_{\ell=0}^{\infty} \left[(\ell+1) \frac{R^{2\ell+1}}{R^{\ell-2}} \right] A_\ell P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} \left(\ell R^{\ell-1}\right) A_\ell P_\ell(\cos\theta)$$

$$\Delta E_\perp\Big|_{r=R}(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) R^{\ell-1} A_\ell P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\epsilon_0}$$

To find the coefficients $\{A_\ell\}$ use the "orthogonality" of the P_ℓ's

$$\int_0^\pi d\theta \sin\theta \, P_m(\cos\theta)\, \Delta E_\perp(\theta) = \int_0^\pi d\theta \sin\theta \frac{\sigma(\theta)}{\epsilon_0} P_m(\cos\theta)$$

$$\Rightarrow (2m+1) R^{m-1} A_m \left(\frac{2}{2m+1}\right) = \int_0^\pi d\theta \sin\theta \frac{\sigma(\theta)}{\epsilon_0} P_m(\cos\theta)$$

$$\therefore A_m = \frac{1}{2\epsilon_0 R^{m-1}} \int_0^\pi \sigma(\theta) P_m(\cos\theta) \overset{\sin\theta}{\wedge} d\theta \qquad \left(\begin{array}{c}\text{Equation 3.79} \\ \text{in Griffiths}\end{array}\right)$$

For the particular $\sigma(\theta)$ given, $\sigma(\theta) = \begin{cases} \sigma_0 & 0 \le \theta \le \pi/2 \\ -\sigma_0 & \frac{\pi}{2} \le \theta \le \pi \end{cases}$

$$\Rightarrow A_m = \frac{\sigma_0}{2\epsilon_0 R^{m-1}} \left[\int_0^{\pi/2} d\theta \sin\theta \, P_m(\cos\theta) - \int_{\pi/2}^{\pi} d\theta \sin\theta \, P_m(\cos\theta) \right]$$

<u>Aside:</u>

Consider $\int d\theta \sin\theta \, P_m(\cos\theta)$ (indefinite integral)

let $\mu = \cos\theta \Rightarrow d\mu = -d\theta \sin\theta$

$$\Rightarrow \int d\theta \sin\theta \, P_m(\cos\theta) = -\int d\mu \, P_m(\mu)$$

$$\therefore A_m = \frac{\sigma_0}{2\epsilon_0 R^{m-1}} \left[\int_0^1 d\mu \, P_m(\mu) - \int_{-1}^{0} d\mu \, P_m(\mu) \right]$$

Now $P_m(\mu)$ is $\begin{cases}\text{an even function of } \mu \text{ for } m\text{-even} \\ \text{an odd function of } \mu \text{ for } m\text{-odd}\end{cases}$

$$\Rightarrow \int_0^1 d\mu \, P_m(\mu) = \int_{-1}^{0} d\mu \, P_m(\mu) \qquad (m \text{ even})$$

$$\int_0^1 d\mu \, P_m(\mu) = -\int_{-1}^{0} d\mu \, P_m(\mu) \qquad (m \text{ odd})$$

Thus, $A_m = \begin{cases} 0 & m\text{-even} \\ \dfrac{\sigma_0}{\epsilon_0 R^{m-1}} \left(\int_0^1 d\mu \, P_m(\mu) \right) & m\text{-odd} \end{cases}$

Now we must calculate $I_m \equiv \int_0^1 d\mu \, P_m(\mu)$ for $m = 1, 3, 5$

$$I_1 = \int_0^1 d\mu \, P_1(\mu) = \int_0^1 d\mu \, \mu = \frac{\mu^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$I_3 = \int_0^1 d\mu \, P_3(\mu) = \int_0^1 \frac{1}{2}(5\mu^3 - 3\mu)\, d\mu = \left[\frac{5\mu^4}{8} - \frac{3\mu^2}{4} \right]_0^1 = -\frac{1}{8}$$

$$I_5 = \int_0^1 d\mu \, P_5(\mu) = \int_0^1 \frac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu)\, d\mu = \left[\frac{63}{48}\mu^6 - \frac{70\mu^4}{32} + \frac{15\mu^2}{16} \right]_0^1$$
$$= \frac{1}{16}$$

$$\therefore \; A_m = \left\{ A_1 = \frac{\sigma_0}{2\epsilon_0}, \; A_3 = -\frac{\sigma_0}{8\epsilon_0 R^2}, \; A_5 = \frac{\sigma_0}{16\epsilon_0 R^4}, \ldots \right\}$$

Thus, the final solution is

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \, r^\ell \, P_\ell(\cos\theta) \qquad r \le R$$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell \, \frac{1}{r^{\ell+1}} \, P_\ell(\cos\theta) \qquad r \ge R$$

$$B_\ell = A_\ell R^{2\ell+1}$$

$$\Rightarrow \boxed{\begin{aligned} V_{in}(r,\theta) &= \frac{\sigma_0 R}{\epsilon_0} \left\{ \frac{1}{2}\left(\frac{r}{R}\right) P_1(\cos\theta) - \frac{1}{8}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \frac{1}{16}\left(\frac{r}{R}\right)^5 P_5(\cos\theta) + \cdots \right\} \\ V_{out}(r,\theta) &= \frac{\sigma_0 R}{\epsilon_0} \left\{ \frac{1}{2}\left(\frac{R}{r}\right)^2 P_1(\cos\theta) - \frac{1}{8}\left(\frac{R}{r}\right)^4 P_3(\cos\theta) + \frac{1}{16}\left(\frac{R}{r}\right)^6 P_5(\cos\theta) + \cdots \right\} \end{aligned}}$$

Define expansion coefficients (set R=σ0=ε0=1)

```
A[1]=1/2
A[3]=-1/8
A[5]=1/16
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Define the Potential
(V in units of (σ0*R)/ε0, r in units of R, u=cosθ)

Inside the sphere

```
Vin[r_,u_] = Sum[A[l]*r^l*LegendreP[l,u],{l,1,5,2}]
(* odd terms only, l=1,3,5  *)
```

Out[10]=

$$\frac{r\ u}{2} - \frac{r^3\ (-3\ u + 5\ u^3)}{16} + \frac{r^5\ (15\ u - 70\ u^3 + 63\ u^5)}{128}$$

Outside the sphere

In[12]:=

```
Vout[r_,u_] = Sum[A[l]/r^(l+1)*LegendreP[l,u],{l,1,5,2}]
```

Out[12]=

$$\frac{u}{2\ r^2} - \frac{-3\ u + 5\ u^3}{16\ r^4} + \frac{15\ u - 70\ u^3 + 63\ u^5}{128\ r^6}$$

General

In[13]:=

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V[r_,u_] := If[r<1, Vin[r,u], Vout[r,u]]
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Plot for r=0 to 2R and over all theta

In[14]:=

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Plot3D[V[r,u], {r,0,2}, {u,-1,1}]
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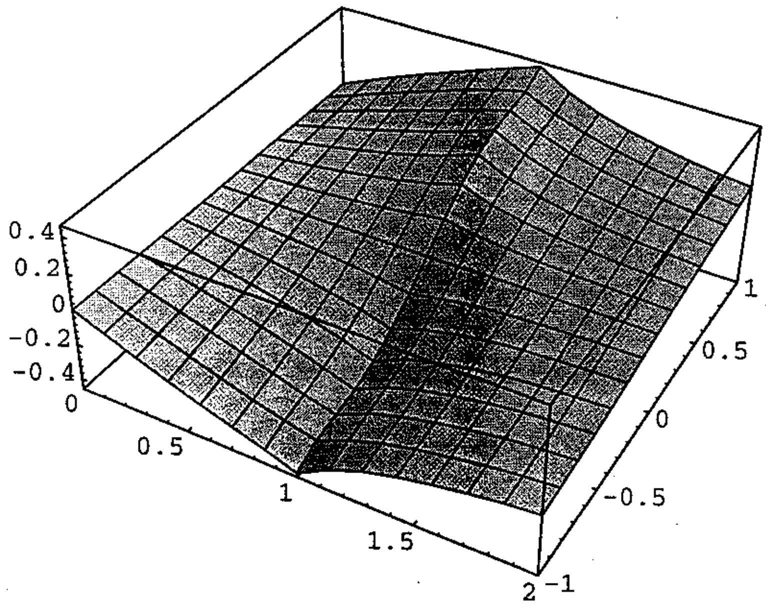

Out[14]=

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-SurfaceGraphics-

(* Notice that the potential is everywhere continuous,
   but its gradient is discontinous in the r-direction
   at the surface of the sphere  *)
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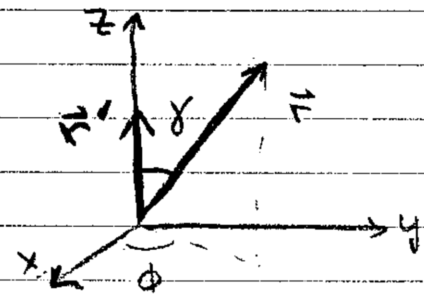
Problem 4

Consider the function $V(\vec{r}) = \frac{1}{|\vec{r}-\vec{r}\,'|}$. Obviously $\nabla^2 V = 0$ for $\vec{r} \neq \vec{r}\,'$ (this is just the potential of a point charge w/o constants)

(a) Let us choose a coordinate system with the z-axis along $\vec{r}\,'$



Here γ is the angle between $\vec{r}$ and $\vec{r}\,'$. In spherical coor the coordinates of $\vec{r}$ are (r, γ, ϕ), i.e. γ is the "θ" coordinate.

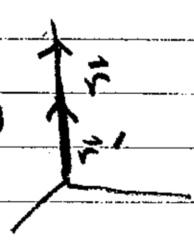
Now $V(\vec{r}) = \frac{1}{\sqrt{(\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$

So $\nabla^2 V = 0$, and V is independent of ϕ, so in general V can be written

$$V(r,\gamma) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\gamma)$$

Now we need to find $\{A_\ell\}$ and $\{B_\ell\}$ for $r > r'$

We can do this by using the b.c. at $\gamma = 0$ (see lecture, March 3)

$\gamma = 0$ 

$$V(r,0) = \frac{1}{r-r'}$$

$$= \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) \quad \left(\text{since } P_\ell(1) = 1\right)$$

Now expand $\frac{1}{r-r'}$ in a power series in r

Since $\frac{r}{r'} < 1$ $\quad \frac{1}{r-r'} = \frac{1}{r}\left(\frac{1}{1-\frac{r'}{r}}\right) = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell}$

(Recall the geometric series: $\sum_{\ell=0}^{\infty} a^{\ell} = \frac{1}{1-a}$)

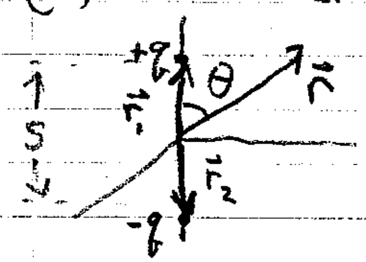
$$\therefore\ V(r,0) = \sum_{\ell=0}^{\infty}\frac{r'^{\ell}}{r^{\ell+1}} = \sum_{\ell=0}^{\infty}\left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}}\right)$$

$$\Rightarrow A_{\ell} = 0, \qquad B_{\ell} = (r')^{\ell}$$

Thus, in general, for $r > r'$

$$\boxed{V(r,\gamma) = \sum_{\ell=0}^{\infty}\frac{(r')^{\ell}}{r^{\ell+1}} P_{\ell}(\cos\gamma)}$$

(b) Consider now the two point charge system



The exact potential is at $\vec{r}$

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\vec{r}-\vec{r}_1|} - \frac{1}{|\vec{r}-\vec{r}_2|}\right\}$$

where $\vec{r}_1 = \frac{s}{2}\hat{z}$, $\vec{r}_2 = -\frac{s}{2}\hat{z}$

Now from the result to part (a), with $\gamma = \theta$, $r' = \frac{s}{2}$

for $r > \frac{s}{2}$ $\quad \frac{1}{|\vec{r}-\vec{r}_1|} = \sum_{\ell=0}^{\infty}\frac{|\vec{r}_1|^{\ell}}{r^{\ell+1}} P_{\ell}(\cos\theta) = \sum_{\ell=0}^{\infty}\frac{\left(\frac{s}{2}\right)^{\ell}}{r^{\ell+1}} P_{\ell}(\cos\theta)$

The angle between $\vec{r}_2$ and $\vec{r}$ is $\pi - \theta = \gamma$
and $\cos(\pi-\theta) = -\cos\theta$.

Also, $P_{\ell}(-x) = (-1)^{\ell} P_{\ell}(x)$ since P_{ℓ} is $\begin{cases}\text{symmetric } \ell\text{-even}\\ \text{anti-sym } \ell\text{-odd}\end{cases}$

(Next Page)

Thus

$$\frac{1}{|\vec{r}-\vec{r}_2|} = \sum_{\ell=0}^{\infty} \frac{|\vec{r}_2|^\ell}{r^{\ell+1}} P_\ell(\cos(\pi-\theta)) \qquad (r > s)$$

$$= \sum_{\ell=0}^{\infty} (-1)^\ell \frac{\left(\frac{s}{2}\right)^\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

The potential for $r > s$ is

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{s^\ell}{r^{\ell+1}} \frac{(1-(-1)^\ell)}{2^\ell} P_\ell(\cos\theta)$$

$$\boxed{V(r,\theta) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=1,3,5}^{\infty} \left(\frac{q s^\ell}{2^{\ell-1}}\right) \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)}$$

Aside the factor $\frac{qs^\ell}{2^{\ell-1}}$ are the multipole moments this charge distribution

$\ell=1$: dipole moment $\quad qs \equiv p$

$\ell=3$: octapole " $\quad \frac{qs^3}{4} = Q^{(3)}$

Up to order 3

$$V(r,\theta) = \underbrace{\frac{1}{4\pi\epsilon_0} \frac{p}{r^2} \cos\theta}_{\text{Dipole Potential}} + \underbrace{\frac{1}{4\pi\epsilon_0} \frac{Q^{(3)}}{r^3} P_3(\cos\theta)}_{\text{Octapole Potential}}$$

(Higher order correction)