# Physics 405

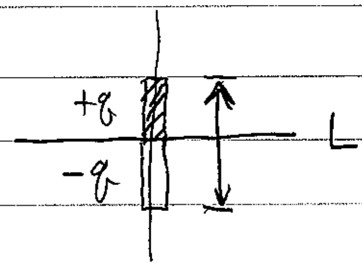
## Problem Set # 8: Solutions

### Problem 1:

The two charge distributions are azimuthally symmetric. Thus the $l^{th}$ order multipole moment can be found by

$$Q^{(l)} = \int d^3r \, \rho(\vec{r}) \, r^l P_l(\cos\theta)$$

(a)




In the case we can determine the $l=0,1,2$ moments by inspection, without doing any integrals

- $\boxed{Q^{(0)} = 0}$ since charge distribution is neutral.
- $Q^{(1)} = +q\langle z\rangle_+ - q\langle z_-\rangle = +q\left(\frac{L}{4}\right) - q\left(-\frac{L}{4}\right)$

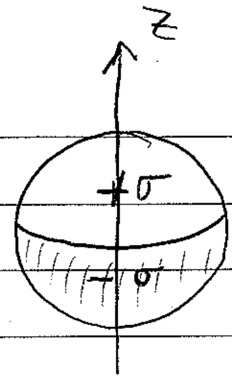
$\Rightarrow \boxed{Q^{(1)} = q\,L/2}$

- $\boxed{Q^{(2)} = 0}$ Since all even moments vanish

Generally, $$Q^{(l)} = \int_0^{L/2} dr \underset{\text{line charge}}{(+\lambda)} r^l \underbrace{P_l(\cos 0^\circ)}_{=1} + \int_0^{L/2} dr\,(-\lambda)\, r^l \underbrace{P_l(\cos\pi)}_{(-1)^l}$$

$$= \frac{2q}{L}\int_0^{L/2} r^l\,dr\,\left(1 + (-1)^{l+1}\right) = 0 \quad l \text{ even}$$

(b)




$$Q^{(\ell)} = \int d^3r\, \rho(\vec{r})\, r^\ell P_\ell(\cos\theta)$$

$$\int \underbrace{2\pi R^2 d(\cos\theta)}_{\text{surface area element}}\ \underbrace{\sigma(\theta)}_{\text{surface charge density}}\ R^\ell P_\ell(\cos\theta)$$

here $\sigma(\theta) = \begin{cases} +\sigma & 0 < \theta < \pi/2 \\ -\sigma & \pi < \theta < \pi/2 \end{cases}$

or $\sigma(\cos\theta) = \begin{cases} +\sigma & 1 < \cos\theta < 0 \\ -\sigma & -1 < \cos\theta < 0 \end{cases}$

$$Q^{(\ell)} = 2\pi R^{\ell+2} \int_{-1}^{+1} d\mu\ \sigma(\mu)\, P_\ell(\mu)$$

We did this integral in Griffiths 3.22

Since $P_\ell(-\mu) = (-1)^\ell P_\ell(\mu)$

$$Q^{(\ell)} = \begin{cases} 4\pi R^{\ell+2}\sigma \int_0^1 d\mu\ P_\ell(\mu) & \ell \text{ odd} \\ 0 & \ell \text{ even} \end{cases}$$

$\Rightarrow \boxed{Q^{(0)} = Q^{(2)} = 0}$

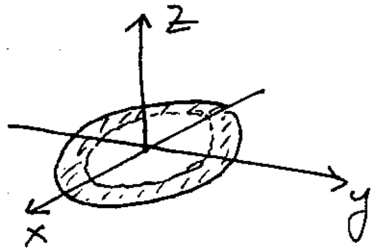
$$Q^{(1)} = 4\pi\sigma R^3 \int_0^1 d\mu\ P_1(\mu) = 4\pi\sigma R^3 \int_0^1 d\mu\ \mu = \boxed{2\pi\sigma R^3}$$

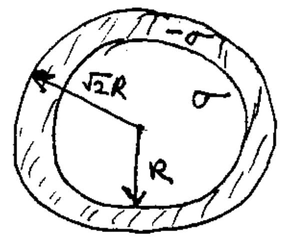
$\boxed{Q^{(1)} = 2\pi\sigma R^3}$

Thus to order $\frac{1}{r^2}$, the Potential is

$$V(r) \cong \frac{1}{4\pi\epsilon_0} \frac{Q^{(1)}}{r^2}\cos\theta = \boxed{\frac{\sigma_0 R}{2\epsilon_0}\left(\frac{R}{r}\right)^2 \cos\theta}$$ as before

Problem 2




(a) According to the multiple expansion

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0} \sum_\ell \frac{Q^{(\ell)}}{r^{\ell+1}} P_\ell(\cos\theta)$$

where $Q^{(\ell)} = \int d^3r' \, (r')^\ell \, P_\ell(\cos\theta') \, \rho(r',\theta')$

For the charge distribution at hand, all of the charge is at $\theta' = \pi/2$ on a surface

$$\Rightarrow Q^{(\ell)} = \int dA' \, (r')^\ell \, P_\ell(\cos(\tfrac{\pi}{2})) \, \sigma(r')$$

where $dA' = 2\pi r' \, dr'$ (area element, no $\phi'$-dep)

$$\sigma(r') = \begin{cases} +\sigma & 0 \le r \le R \\ -\sigma & R \le r \le \sqrt{2}R \\ 0 & r > \sqrt{2}R \end{cases}$$

$$\therefore Q^{(\ell)} = 2\pi \int dr' \, (r')^{\ell+1} \, P_\ell(0) \, \sigma(r')$$

Remember $P_\ell(0) = 0$ if $\ell$ is odd

$Q^{(0)} = Q_{Net}$ (no need to do an integral here)

$= \sigma$ (Area inner disk) $- \sigma$ (Area of annulus)

$= \sigma \left( \pi R^2 - \left( \pi (\sqrt{2} R)^2 - \pi R^2 \right) \right)$

$= \sigma \left( -2\pi R^2 + \pi R^2 + \pi R^2 \right) = 0$ (neutral)

$Q^{(1)} = 0$ ~~since~~ ($\ell$ odd) (we could have guessed this since $Q^{(1)} = p_z$)

$$Q^{(2)} = 2\pi \int dr' \, (r')^3 \, P_2(0) \, \sigma(r')$$

$$= -\pi \int dr' \, (r')^3 \, \sigma(r') \quad \left( \text{since } P_2(0) = -\tfrac{1}{2} \right)$$

$$= \pi\sigma \left( -\int_0^R (r')^3 dr' + \int_R^{\sqrt{2} R} (r')^3 dr' \right)$$

$$= \pi\sigma \left( -\frac{r'^4}{4} \Big|_0^R + \frac{r'^4}{4} \Big|_R^{\sqrt{2} R} \right)$$

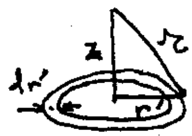
$$= \pi\sigma R^4 \left( -\frac{R^4}{4} + \frac{4R^4}{4} - \frac{R^4}{4} \right)$$

$$= \frac{\pi \sigma R^4}{2} \quad (\text{Units: Charge} \cdot \text{length}^2)$$

$$\therefore \; V(r,\theta) = \frac{1}{4\pi\epsilon_0} \frac{Q^{(2)}}{r^3} P_2(\cos\theta)$$

$$\Rightarrow \boxed{V(r,\theta) = \frac{\sigma}{8\epsilon_0} \frac{R^4}{r^3} P_2(\cos\theta)}$$

(b). By direct integration, the potential along the $z$-axis is found by adding up the contribution of rings of charge of radius $r'$ and thickess $dr'$




$$dV = \frac{1}{4\pi\epsilon_0}\frac{\sigma(r')\,dA'}{\mathscr{r}}, \quad dA' = 2\pi r' dr'$$

$$= \frac{1}{2\epsilon_0}\frac{\sigma(r')dr'}{\sqrt{r'^2+z^2}}$$

$$\Rightarrow V(z) = \int dV = \frac{1}{2\epsilon_0}\int \frac{dr'\,\sigma(r')}{\sqrt{r'^2+z^2}}$$

$$= \frac{\sigma}{2\epsilon_0}\left[\int_0^R \frac{dr'}{\sqrt{r'^2+z^2}} - \int_R^{\sqrt{2}R}\frac{dr'}{\sqrt{r'^2+z^2}}\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[\sqrt{r'^2+z^2}\,\Big|_0^R - \sqrt{r'^2+z^2}\,\Big|_R^{\sqrt{2}R}\right]$$

$$\Rightarrow V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - z - \sqrt{z^2+2R^2} + \sqrt{z^2+R^2}\right)$$

$$\Rightarrow \boxed{V(z) = \frac{\sigma}{2\epsilon_0}\left(2\sqrt{z^2+R^2} - \sqrt{z^2+2R^2} - z\right)}$$

(c) Since $\rho$ is azimulthally symmetric, we know $V$ is independent of $\phi$, and therefore outside the charge distribution

$$V(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

for $r > \sqrt{2}\,R$ we have the b.c. $V \to 0$ and $r \to \infty$

$$\Rightarrow \quad A_\ell = 0 \quad \forall \ell$$

$$\Rightarrow \quad V(r,\theta) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

We also have the b.c. $V(r,\theta=0)$ (along z-axis)

$$V(r,\theta=0) = \frac{\sigma}{2\epsilon_0}\left(2\sqrt{r^2+R^2} - \sqrt{r^2+2R^2} - r\right)$$

To find the expansion coefficients, $B_\ell$, expand the above expression in powers of $\frac{R}{r} << 1$

$$V(r,\theta=0) = \frac{\sigma}{2\epsilon_0} r\left(2\left(1+\frac{R^2}{r^2}\right)^{1/2} - \left(1+\frac{2R^2}{r^2}\right)^{1/2} - 1\right)$$

$$\approx \frac{\sigma}{2\epsilon_0} r \left\{ 2\left(1+\frac{R^2}{2r^2} - \frac{1}{8}\left(\frac{R^2}{r^2}\right)^2\right) - \left(1+\frac{1}{2}\frac{2R^2}{r^2} - \frac{1}{8}\left(\frac{2R^2}{r^2}\right)^2\right) - 1 \right\}$$

here I used $(1+\delta)^n \approx 1 + n\delta + \frac{n(n-1)}{2}\delta^2$ for $\delta < 1$

$$\therefore\ V(r, \theta = 0) \approx \frac{\sigma}{2\epsilon_0} r \left( 2 + \frac{R^2}{r^2} - \frac{1}{4}\frac{R^4}{r^4} - 1 - \frac{R^2}{r^2} + \frac{1}{2}\frac{R^4}{r^4} - 1 \right)$$

$$= \frac{\sigma}{2\epsilon_0} r \left( \frac{R^4}{4r^4} \right) = \frac{\sigma R^4}{8\epsilon_0} \frac{1}{r^3}$$

The general expansion is

$$V(r, \theta) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow V(r, \theta = 0) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}} P_\ell(0) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}}$$

$$= \frac{B_0}{r} + \frac{B_1}{r^2} + \frac{B_2}{r^3} + \cdots$$

$$\Rightarrow B_0 = B_1 = 0 \qquad B_2 = \frac{\sigma R^4}{8\epsilon_0}$$

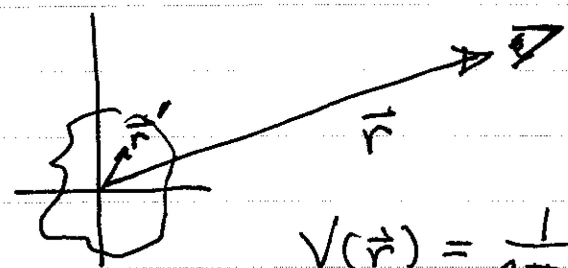
$\therefore$ Up to order $1/r^3$

$$\boxed{V(r, \theta) = \frac{\sigma R^4}{8\epsilon_0} \frac{1}{r^3} P_2(\cos\theta)}$$

(as in part (a))

Problem 3

We start with the exact expression for the potential given a localized charge distribution




$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}$$

From our previous studies, for $r > r'$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r'^{\ell}}{r^{\ell+1}} P_\ell(\hat{r}\cdot\hat{r}')$$

Legendre polynomial

where $\hat{r}\cdot\hat{r}' = \cos\gamma \leftarrow$ angle between $\hat{r}$ and $\hat{r}'$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\int d^3r'\,(r')^{\ell}\,P_\ell(\hat{r}\cdot\hat{r}')\,\rho(\vec{r}\,')$$

Recall: $P_0(\mu) = 1$, $P_1(\mu) = \mu$, $P_2(\mu) = \frac{3}{2}\mu^2 - \frac{1}{2}$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\Big[\frac{1}{r}\int d^3r'\,\rho(\vec{r}\,') + \frac{\hat{r}}{r^2}\cdot\int d^3r'\,(r'\hat{r}')\,\rho(\vec{r}\,')$$

$$+ \frac{1}{r^3}\int d^3r'\left[\frac{3}{2}(\hat{r}\cdot\hat{r}')^2 - \frac{1}{2}\right]\rho(\vec{r}\,') + \dots\Big]$$

We recognize the first two terms as the monopole and dipole contributions

$$V^{(0)}_{(\vec{r})} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad : \quad Q = \int d^3r' \, \rho(\vec{r}')$$

$$V^{(1)}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2} \quad : \quad \vec{p} = \int d^3r' \, \vec{r}' \rho(\vec{r}')$$

The last term is the quadrupole:

$$V^{(2)}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r^3}\sum_{i,j=1}^{3} \hat{r}_i \hat{r}_j Q_{ij}\right)$$

where $i,j$ label the 3-Cartesian directions $1=x, 2=y, 3=z$

The Quadrupole tensor (matrix)

$$Q_{ij} = \int \frac{1}{2}(3r_i' r_j' - (r')^2 \delta_{ij}) \, \rho(\vec{r}') \, d^3r'$$

Check: $$\sum_{ij} \hat{r}_i Q_{ij} \hat{r}_j = \int \sum_{ij} \frac{1}{2}\left(3(r_i'\hat{r}_i')(r_j'\hat{r}_j) - (r')^2 \delta_{ij} \hat{r}_i' \hat{r}_j'\right) \rho(\vec{r}') \, d^3r'$$

$$= \int r'^2 \left(\frac{3}{2}(\hat{r}\cdot\hat{r}')^2 - \frac{1}{2}\right) \rho(\vec{r}') \, d^3r'$$

$$= \int r'^2 P_2(\hat{r}\cdot\hat{r}') \, \rho(\vec{r}') \, d^3r' \quad \checkmark$$

(b) Given $Q_{ij} = \int \frac{1}{2}\left(3r_i'r_j' - (r'^2)\delta_{ij}\right)\rho(\vec{r}')\,d^3r'$

Note: $\boxed{Q_{ij} = Q_{ji}}$ since $r_i'r_j' = r_j'r_i'$ (commute)

$\delta_{ij} = \delta_{ji}$

Trace: Sum diagonal matrix elements

$$\sum_i Q_{ii} = \int \frac{1}{2}\left(3\underbrace{\left(\sum_i r_i'r_i'\right)}_{=|\vec{r}'|^2} - |\vec{r}'|^2 \underbrace{\sum_i \delta_{ii}}_{=3}\right)\rho(\vec{r}')\,d^3r'$$

$$\Rightarrow \boxed{Q_{xx} + Q_{yy} + Q_{zz} = 0}$$

(c) For azimuthal symmetry $Q_{xx} = Q_{yy}$

$$\Rightarrow \boxed{Q_{zz} = -2Q_{xx}}$$

Also, for azimuthal symmetry $\rho(r,\theta)$ (independent of azimuthal angle $\phi$)

Off-diagonal: $i \neq j$ $\quad Q_{ij} = \frac{3}{2}\int r_i' r_j' \underbrace{\rho(\vec{r}')}_{\rho(r,\theta)}\ \underbrace{d^3r'}_{2\pi\, d(\cos\theta)\, r^2 dr}$

But integral $\int xy\, \rho(r,\theta)\, d^3r = \int xz\, \rho(r,\theta)\, d^3r$

$$= \int yz\, \rho(r,\theta)\, d^3r = 0$$

Since integral over $\phi$ will be zero

(d) Given $Q_{ij} = Q_{zz} \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

~~for azimuthal symmetry~~

$\Rightarrow V_{quad} = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r^3} \sum_{ij} \hat{r}_i Q_{ij} \hat{r}_j$

Aside: $\sum_{ij} \hat{r}_i Q_{ij} \hat{r}_j = (\hat{r}_x)^2 Q_{xx} + (\hat{r}_y)^2 Q_{yy}$
$+ (\hat{r}_z)^2 Q_{zz}$

$= Q_{zz} \left[ (\hat{r}_z)^2 - \frac{(\hat{r}_x)^2 + (\hat{r}_y)^2}{2} \right]$

<u>Aside</u> $\hat{r} = \frac{\vec{r}}{r} = \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y}$
$+ \cos\theta\, \hat{y}$

$\hat{r}_x^2 + \hat{r}_y^2 + \hat{r}_z^2 = 1 \Rightarrow \hat{r}_x^2 + \hat{r}_y^2 = 1 - \hat{r}_z^2$

$\Rightarrow = Q_{zz} \left( \frac{3}{2} (\hat{r}_z)^2 - \frac{1}{2} \right)$

$= Q_{zz} \left( \frac{3}{2} \cos^2\theta - \frac{1}{2} \right)$

$= Q_{zz}\, P_2(\cos\theta)$ ✓

Phew!