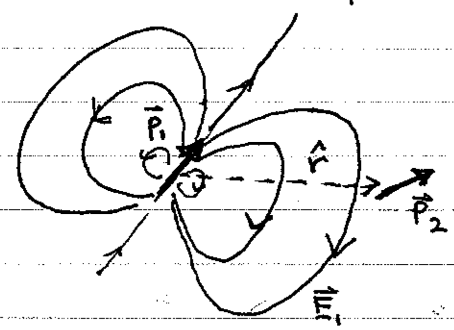
Physics 405

Problem Set #9 Solutions

Prob 1: We seek the potential energy of interaction between two dipoles $\vec{p}_1$ and $\vec{p}_2$



The electric field due to $\vec{p}_1$ exerts a force on dipole $\vec{p}_2$

$$\vec{F}_{21} = (\vec{p}_2 \cdot \vec{\nabla})\,\vec{E}_1$$

This force is then the gradient of a potential energy

$$U_{21} = -\,\vec{p}_2 \cdot \vec{E}_1$$

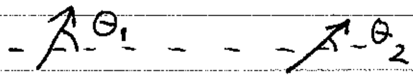
Now, according to Prob #4 of P.S. #8 we found the field of a dipole

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[3(\vec{p}_1\cdot\hat{r})\,\hat{r} - \vec{p}_1\right]$$

Thus
$$\boxed{U_{21} = +\frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[\vec{p}_1\cdot\vec{p}_2 - 3\,(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\right]}$$

Note that this depends on both the ~~difference~~ distance between the dipoles as well as the angle between them

If we define $\theta_{1,2}$ ~~and~~ as the angle between $\vec{p}_{1,2}$ and and vector between them



$$\Rightarrow\quad U_{21} = \frac{1}{4\pi\epsilon_0}\,\frac{p_1 p_2}{r^3}\left[\cos(\theta_1-\theta_2) - 3\cos\theta_1\cos\theta_2\right]$$

Note: $|\vec{F}_{21}| \propto \frac{1}{r^4}$ (very weak compared to force between point charges)

Energy for particular configurations

(i) $\vec{p}_1 \uparrow \underset{\leftarrow d \rightarrow}{} \uparrow \vec{p}_2$

$$\boxed{U_{(i)} = +\frac{1}{4\pi\epsilon_0}\frac{p_1 p_2}{d^3}}$$

$\theta_1 = \theta_2 = 90°$

(ii) $\vec{p}_1 \downarrow \underset{\leftrightarrow d}{} \uparrow \vec{p}_2$

$$\boxed{U_{(ii)} = -\frac{1}{4\pi\epsilon_0}\frac{p_1 p_2}{d^3}}$$

$\theta_1 = -90°$ $\theta_1 = 90°$

(iii) $\overset{\vec{p}_1}{\longrightarrow}$ $\overset{\vec{p}_2}{\longrightarrow}$ $|\leftarrow d \rightarrow|$

$$\boxed{U_{(iii)} = -\frac{1}{4\pi\epsilon_0}\frac{2p_1 p_2}{d^3}}$$

$\theta_1 = \theta_2 = 0°$

(iv) $\overset{\vec{p}_1}{\longleftarrow}$ $\overset{\vec{p}_2}{\longrightarrow}$ $|\leftarrow d \rightarrow|$

$$\boxed{U_{(iv)} = +\frac{1}{4\pi\epsilon_0}\frac{2p_1 p_2}{d^3}}$$

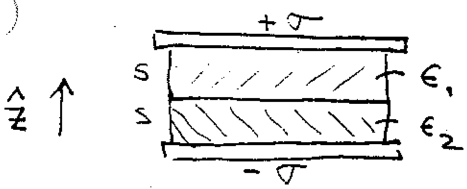
$\theta_1 = 180°$ $\theta_2 = 0°$

Thus the lowest energy configuration is $\overset{\vec{p}_1}{\longrightarrow}$ $\overset{\vec{p}_2}{\longrightarrow}$

For a 1-D line of dipoles, their mutual interacts tends to align them. This is the origin of a permanently polarized material (with no external field).

Of course thermal fluctuations and stray fields (due to other free charges) tends to destroy this alignment.

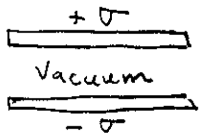
Prob. 2 Griffiths 4.18



$$\epsilon_1 = \epsilon_0(1+\chi_{e,1}) = \epsilon_0 K_1$$
$$K_1 = 2$$
$$\epsilon_2 = \epsilon_0 K_2 \,, \quad K_2 = \frac{3}{2}$$

(a) For a linear dielectric $\vec{D}$ is completely determined by the free charges σ

$\vec{D} = \epsilon_0 \vec{E}_{vac}$ where $\vec{E}_{vac}$ is the field associated with the free charges with no dielectrics



$$\vec{E}_{vac} = -\frac{\sigma}{\epsilon_0}\hat{z} \quad \text{(ignoring fringing fields)}$$

$$\therefore \boxed{\vec{D} = -\sigma\hat{z}}$$

(b) For a linear dielectric $\vec{D} = \epsilon\vec{E}$ or $\vec{E} = \frac{\vec{D}}{\epsilon}$

Slab 1: $\vec{E}_1 = \frac{\vec{D}}{\epsilon_1} = \frac{\vec{D}}{\epsilon_0 K_1} = \boxed{-\frac{1}{2}\frac{\sigma}{\epsilon_0}\hat{z}}$

Slab 2: $\vec{E}_2 = \frac{\vec{D}}{\epsilon_2} = \frac{\vec{D}}{\epsilon_0 K_2} = \boxed{-\frac{2}{3}\frac{\sigma}{\epsilon_0}\hat{z}}$

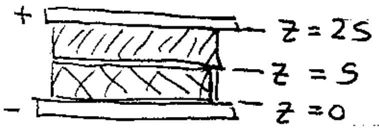
(c) Again for linear dielectric $\vec{P} = \epsilon_0\chi_e\vec{E} = \epsilon_0(K-1)\vec{E}_1$

Slab 1: $\vec{P}_1 = \epsilon_0(K_1-1)\vec{E}_1 = \boxed{-\frac{1}{2}\sigma\hat{z}}$

Slab 2: $\vec{P}_2 = \epsilon_0(K_2-1)\vec{E}_2 = \epsilon_0\left(\frac{3}{2}-1\right)\left(-\frac{2}{3}\frac{\sigma}{\epsilon_0}\hat{z}\right)$

$$= \boxed{-\frac{1}{3}\sigma\hat{z}}$$

(d) Potential difference between (+) and (-) plates



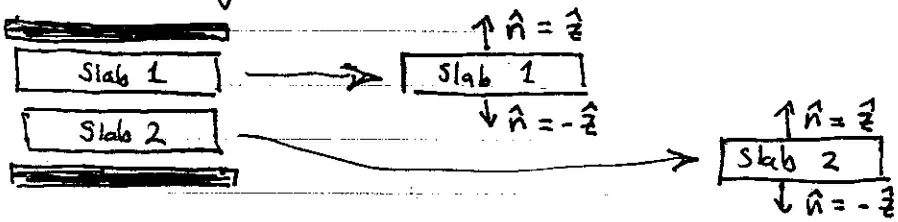
$$V_{+-} = -\int_{-}^{+} \vec{E}\cdot d\vec{\ell}$$
$$= -\int_0^s E_2\, dz - \int_s^{2s} E_1\, dz$$

$$\therefore\ V_{+-} = -E_2 s - E_1 s = \frac{1}{2}\frac{\sigma}{\epsilon_0} s + \frac{2}{3}\frac{\sigma}{\epsilon_0} s$$

$$\Rightarrow \boxed{V_{+-} = \frac{7}{6}\frac{\sigma}{\epsilon_0} s}$$

Note: without dielectrics $V_{+-}^{vac} = \frac{\sigma}{\epsilon_0}(2s) > V_{+-}^{w/\ dielectric}$

(e) Bound charge exists at the surface of each dielectric



Bound surface charge: $\sigma_b = \hat{n}\cdot\vec{P}$

- Top of slab 1 $(z = 2s)$ $\quad \sigma_b = \hat{z}\cdot\vec{P}_1 = \boxed{-\frac{1}{2}\sigma}$
- Between slabs $(z = s)$ $\quad \sigma_b = -\hat{z}\cdot\vec{P}_1 + \hat{z}\cdot\vec{P}_2 = \hat{z}\cdot(\vec{P}_2 - \vec{P}_1)$
$$= \boxed{+\frac{1}{6}\sigma}$$
- Bottom of slab 2 $(z = 0)$ $\quad \sigma_b = -\hat{z}\cdot\vec{P}_2$
$$= \boxed{\frac{1}{3}\sigma}$$

(f) The electric field is due to the total charge (bound and free)

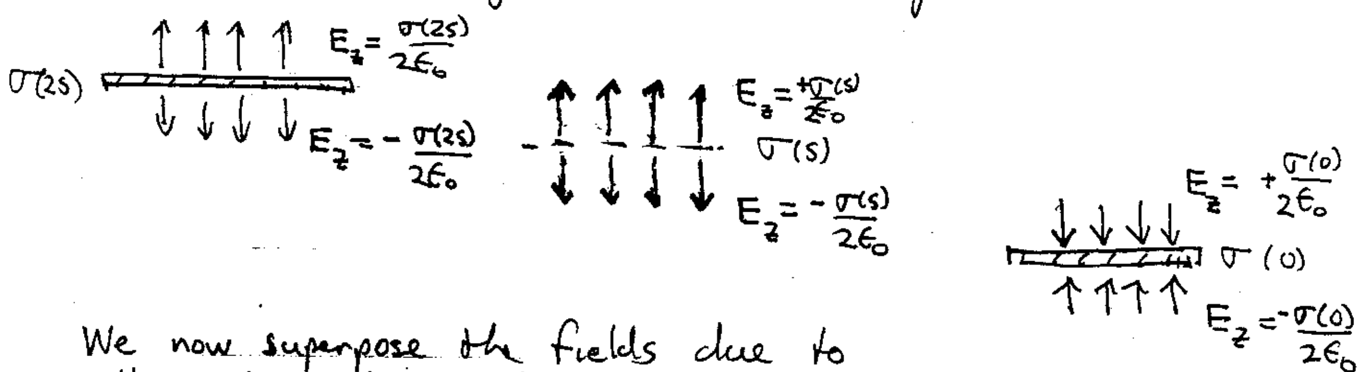
Total charge

$$\sigma(2s) = \sigma_{free} + \sigma_{bound} = \sigma - \tfrac{1}{2}\sigma = \frac{\sigma}{2}$$

$$\sigma(s) = \sigma_{bound} = +\frac{\sigma}{6}$$

$$\sigma(0) = \sigma_{free} + \sigma_{bound} = -\sigma + \frac{\sigma}{3} = -\frac{2\sigma}{3}$$

Now we find the field of each sheet of charge (individually)



We now superpose the fields due to all of these charges

$$z > 2s: \quad \vec{E}_{above} = \left(\frac{\sigma(2s) + \sigma(s) + \sigma(0)}{2\epsilon_0}\right)\hat{z} = \frac{\frac{\sigma}{2} + \frac{\sigma}{6} - \frac{2\sigma}{3}}{2\epsilon_0} = 0 \checkmark$$

ide 1

$$s < z < 2s: \quad \vec{E}_1 = \left(\frac{-\sigma(2s) + \sigma(s) + \sigma(0)}{2\epsilon_0}\right)\hat{z} = \frac{-\frac{\sigma}{2} + \frac{\sigma}{6} - \frac{2\sigma}{3}}{2\epsilon_0} = -\frac{\sigma}{2\epsilon_0} \checkmark$$

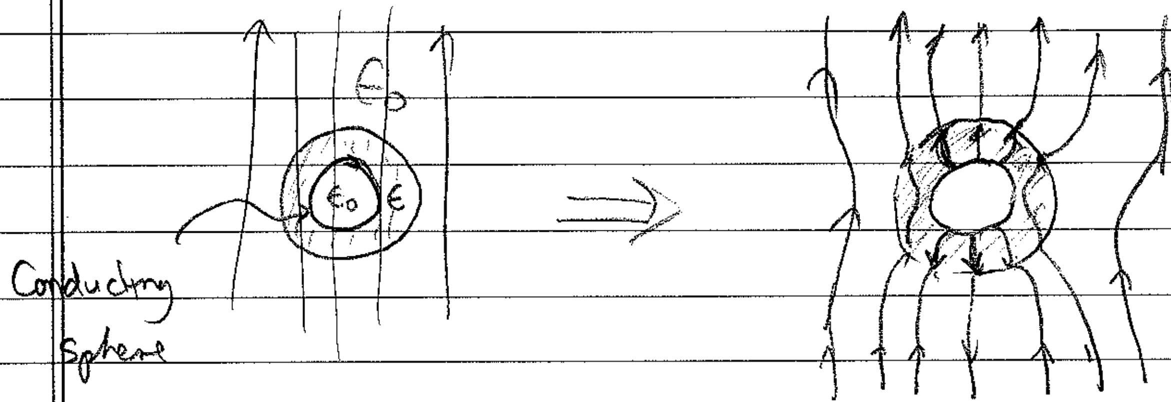
side slab 2

$$0 < z < s: \quad \vec{E}_2 = \left(\frac{-\sigma(2s) - \sigma(s) - \sigma(0)}{2\epsilon_0}\right)\hat{z} = \left(\frac{-\frac{\sigma}{2} - \frac{\sigma}{6} - \frac{2\sigma}{3}}{2\epsilon_0}\right) = -\frac{2\sigma}{3\epsilon_0} \checkmark$$

$$z < 0 \quad \vec{E}_{below} = -\left(\frac{\sigma(2s) + \sigma(s) + \sigma(0)}{2\epsilon_0}\right) = 0 \checkmark$$

Agrees with previous calculations

Problem 3: Griffiths 4.24



This problem is a hybrid of examples 3.8 and 4.7 in the text.

There a three regions

(I) $r < a$ inside conducting sphere

(II) $a < r < b$ inside dielectric shell

(III) $r > b$ free space outside material

From our knowledge of conductors and the nature of the solutions with azimuthal symmetry

$$V^{(I)}(r,\theta) = 0$$

$$V^{(II)}(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l^{(II)}}{r^{l+1}} \right) P_l(\cos\theta)$$

$$V^{(III)}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l^{(III)}}{r^{l+1}} P_l(\cos\theta) \underbrace{- E_0 r \cos\theta}_{\text{asymptote b.c.}}$$

Boundary conditions

Continuity of potential
(i) $V^{(I)}(a,\theta) = V^{(II)}(a,\theta)$

(ii) $V^{(II)}(b,\theta) = V^{(III)}(b,\theta)$

Continuity of normal derivation of $\vec{D} = \epsilon\vec{E}$ (no free charge at $r=b$)
(iii) $\epsilon \frac{\partial V^{(II)}}{\partial r}(b,\theta) = \epsilon_0 \frac{\partial V^{(III)}}{\partial r}(b,\theta)$

Using b.c. (i)

$$\Rightarrow \sum_{\ell} \left(A_\ell a^\ell + \frac{B_\ell^{(II)}}{a^{\ell+1}}\right) P_\ell(\cos\theta) = 0$$

By orthogonality

$$\Longrightarrow A_\ell = -\frac{1}{a^{2\ell+1}} B_\ell^{(II)}$$

$$\Rightarrow V^{(II)}(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell^{(II)}\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{a^{2\ell+1}}\right) P_\ell(\cos\theta)$$

Now we know from our experience with these types of problems (spheres in uniform $\vec{E}$ fields), the only non-vanishing term in the expansion is $\ell = 1$

$$\Rightarrow V^{(II)}(r,\theta) = B_1^{(II)}\left(\frac{1}{r^2} - \frac{r}{a^3}\right)\cos\theta$$

$$V^{(III)}(r,\theta) = \left[-E_0 r + \frac{B_1^{(III)}}{r^2}\right]\cos\theta$$

Using b.c. #'s (ii) and (iii) we can solve for the two unknowns, $B_1^{(II)}$ and $B_1^{(III)}$

b.c. (ii)

$$\Rightarrow B_1^{(II)}\left(\frac{1}{b^2} - \frac{b}{a^3}\right) = -E_0 b + \frac{B_1^{(III)}}{b^2}$$

b.c. (iii)

$$\Rightarrow K B_1^{(II)}\left(\frac{-2}{b^3} - \frac{1}{a^3}\right) = -E_0 - \frac{2B_1^{(III)}}{b^3}$$

$K = \epsilon/\epsilon_0 = \epsilon_r$ (in Griffith)

We thus have two equations, two unknowns. After some algebra we find:

$$\boxed{B_1^{(II)} = \frac{3a^3b^3}{2a^3(K-1)+b^3(K+2)} E_0}$$

$$\boxed{B_1^{(III)} = \frac{b^6(K-1)+a^3b^3(2K+1)}{2a^3(K-1)+b^3(K+2)} E_0}$$

Check $K \to 1$ $B_1^{(II)} = B_1^{(III)} = a^3 E_0$, in agreement with example 3.8

Check $a \ll b$ $B_1^{(II)} \to \frac{3}{K+2} a^3 E_0$

$$B_1^{(III)} \to b^3 \left(\frac{K-1}{K+2}\right) E_0$$

In agreement with Ex. 4.7

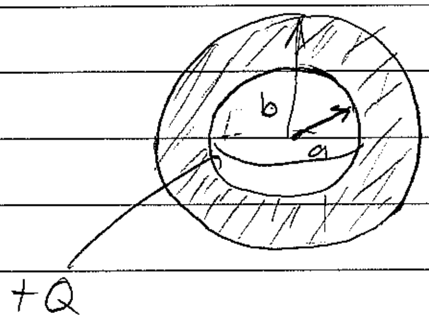
Thus, the field inside the dielectric

$$\boxed{\begin{aligned} E^{(II)}(r,\theta) &= -\frac{\partial V(r,\theta)}{\partial r} \\ &= B_1^{(II)}\left(+\frac{2}{r^3} + \frac{1}{a^3}\right)\cos\theta \end{aligned}}$$

where $\boxed{B_1^{(II)} = \dfrac{3a^3}{K+2+2\left(\frac{a}{b}\right)^3(K-1)}}$

Compare with Eq. 4.49 in Griffiths

Problem 4 Griffiths 4.26



Spherical conductor carries a charge Q, surround by dielectric with susceptibility χ. Find energy of configuration

For charges in the presence of linear dielectrics, the energy of configuration

$$U = \int \frac{\vec{D}\cdot\vec{E}}{2}\, d^3r$$

where $\vec{D}(\vec{r}) = \epsilon(\vec{r})\,\vec{E}(\vec{r}) = \epsilon_0\left(1+\chi(\vec{r})\right)\vec{E}(\vec{r})$

Here $\chi(\vec{r}) = \begin{cases} 0 & r<a \\ \chi & a<r<b \\ 0 & r>b \end{cases}$

To find $\vec{D}$ and then $\vec{E}$, use Gauss's Law

$$\oint \vec{D}\cdot d\vec{a} = Q_{free}^{enc}$$

By symmetry $\vec{D}(\vec{r}) = \hat{r}\, D(r)$

$$\Rightarrow D(r) = \begin{cases} 0 & r<a \\ \dfrac{Q}{4\pi r^2} & r>a \end{cases}$$

$$\Rightarrow E(r) = \frac{D(r)}{\epsilon(r)} = \begin{cases} 0 & r < a \\ \frac{Q}{4\pi\epsilon r^2} & a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2} & r > b \end{cases}$$

$$\Rightarrow U = \int_0^\infty \underbrace{4\pi r^2 dr}_{= d^3r \text{ for spherical symmetry}} \frac{D^2(r)}{2\epsilon(r)} = \frac{Q^2}{8\pi\epsilon}\int_a^b \frac{dr}{r^2} + \frac{Q^2}{8\pi\epsilon_0}\int_b^\infty \frac{dr}{r^2}$$

$$\Rightarrow \boxed{U = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{b}\right]}$$

Check $\chi \to 1$ (vacuum)

$$U \Rightarrow \frac{Q^2}{8\pi\epsilon_0 a}$$ as in Lecture 10

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