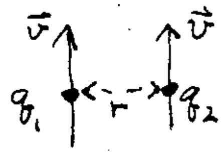
Physics 405, Spring 1996

Problem Set #10 Solutions

Problem 1

(a)



Force on q_1 due to q_2

Electric: $|\vec{F}_E| = q_1 E(\vec{r}_1) = \dfrac{q_1 q_2}{4\pi\epsilon_0 r^2}$ (repulsive)

Magnetic: $v << c$ = speed of light

$$|\vec{F}_B| = q_1 q_2 \frac{\mu_0}{4\pi}\frac{v^2}{r^2}$$

$$\Rightarrow \frac{|\vec{F}_E|}{|\vec{F}_B|} = \frac{1}{\mu_0 \epsilon_0 v^2} = \frac{c^2}{v^2} \qquad \text{since } \mu_0^2\epsilon_0^2 = \frac{1}{c^2}$$

Thus for nonrelativistic charges $|\vec{F}_E| >> |\vec{F}_B|$

(b) The equation of motion of charges in electric and magnetic fields is determined by the Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = m\underset{\text{acceleration}}{\vec{a}} = m\frac{d\vec{v}}{dt}$$

Differentiating with respect to time again $\left(\frac{d\vec{E}}{dt} = \frac{d\vec{B}}{dt} = 0\right)$

$$\Rightarrow \frac{d^2\vec{v}}{dt^2} = \frac{q}{m}\left(\frac{d\vec{v}}{dt} \times \vec{B}\right)$$

Now plug back in for $\frac{d\vec{v}}{dt}$

(Next Page)

$$\Rightarrow \frac{d^2\vec{v}}{dt^2} = \frac{q}{m}\left(\frac{q}{m}(\vec{E} + \vec{v}\times\vec{B})\times\vec{B}\right)$$

$$= \frac{q^2}{m^2}(\vec{E}\times\vec{B}) + \frac{q}{m}(\vec{v}\times\vec{B})\times\vec{B}$$

<u>Aside</u>: $(\vec{v}\times\vec{B})\times\vec{B} = \vec{B}(\vec{v}\cdot\vec{B}) - \vec{v}\,\underbrace{(\vec{B}\cdot\vec{B})}_{=B^2}$

$$\therefore \boxed{\frac{d^2\vec{v}}{dt^2} = -\frac{q^2B^2}{m^2}\vec{v} + \frac{q^2}{m^2}(\vec{E}\times\vec{B}) + \frac{q^2}{m^2}(\vec{B}\cdot\vec{v})\vec{B}}$$

where $\omega_c = \frac{qB}{m}$ (The cylcotron frequency) :

(b) $\vec{E} = E_0\hat{y}$, $\vec{B} = B_0\hat{z}$, $\vec{v}\cdot\vec{B} = 0$

$$\Rightarrow \frac{d^2\vec{v}}{dt^2} = -\omega_c^2\vec{v} + \frac{q^2}{m^2}(\vec{E}\times\vec{B})$$

$$= -\omega_c^2\left(\vec{v} - \frac{\vec{E}\times\vec{B}}{B^2}\right)$$

This is the differential equation for a forced harmonic oscillator (constant driving term)

The general solution some of a "homogeneous" and "particular" solution

$$\vec{v}(t) = \vec{v}_{Hom}(t) + \vec{v}_{Part.}(t)$$

(Next Page)

To find the homogenous solution, set the forcing term to zero

$$\Rightarrow \quad \frac{d^2\vec{v}_{1hom}}{dt^2} = -\omega_c^2\, \vec{v}_{1hom}$$

Each ~~solution~~ component is a simple harmonic oscillator

If the intial conditions are

$$|\vec{v}(t=0)| = v_0 \,, \qquad \frac{\vec{v}(t=0)}{|\vec{v}|} = \cos\phi\, \hat{x} + \sin\phi\, \hat{y}$$

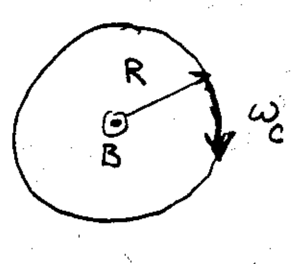
Then $\vec{v}_{1hom}(t) = v_0 \cos(\omega_c t+\phi)\, \hat{x} + v_0 \sin(\omega_c t+\phi)\, \hat{y}$

the homogenous solution is the ~~cyc~~ cycltron motion discussed in class. Integrating with ~~to~~ respect to time gives the position as a function of time

$$\Rightarrow \quad \vec{r}_{Hom}(t) = \frac{v_0}{\omega_c} \sin(\omega_c t+\phi)\, \hat{x} - \frac{v_0}{\omega_c} \cos(\omega_c t+\phi)\, \hat{y}$$

This is a parametric equation to a circle of

$R = \frac{v_0}{\omega_c}$ (the cyclotron radius)



Note, the ~~cyctr~~ cyclotron radius depends of v_0, but the angular velocity ω_c does not

Now we must add the particular solution due to the driving force.

Since the ~~decreasing~~ driving force is time independent, so must be particular solution

$$\Rightarrow \quad 0 = -\omega_c^2\left(\vec{v}_{part} - \frac{\vec{E}\times\vec{B}}{B^2}\right)$$

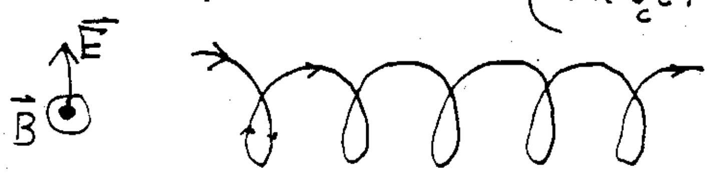
$$\Rightarrow \quad \vec{v}_{part} = \frac{\vec{E}\times\vec{B}}{B^2}$$

Thus the general solution is

$$\boxed{\vec{v}(t) = v_0 \cos(\omega_c t+\phi)\,\hat{x} + v_0 \sin(\omega_c t+\phi)\,\hat{y} + \frac{\vec{E}\times\vec{B}}{B^2}}$$

(c) A sample trajectory is found by t-integration

$$\vec{r}(t) = R\left(\sin(\omega_c t+\phi)\,\hat{x} + \cos(\omega_c t+\phi)\,\hat{y}\right) + \frac{E_0 t}{B_0}\,\hat{x}$$



The electric field accellerates (decellerates) the positive charge moving with (against) t. This changes the local cyclotron radius causing the particle to spiral in the direction of $\vec{E}\times\vec{B}$

In a cyclotron period $T_c = \frac{2\pi}{\omega_c}$, $\vec{r}(t_c) = \frac{E_0 T_c}{B_0}\,\hat{x}$

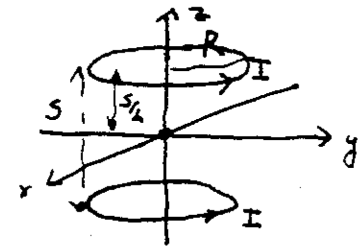
The number cyltron radius the particle drifts in $\hat{x}$-direction

$$\frac{\Delta X(T_c)}{R} = \frac{E_0 T_c}{B_0 R} = \frac{2\pi E_0}{B_0 \omega_0 R} = \frac{2\pi E_0}{B_0 v_0} =$$

$$= 2\pi\left(.01\,\tfrac{V}{m}\right)/\left(1T \cdot 10\,\tfrac{m}{s}\right) = \boxed{2\pi\times 10^{-3}\ m}$$

Problem 3 Griffiths 5.59

(a) Two current loops



Know the magnetic field on the axis of a single loop, the field for two loops is found by superposition

$$\boxed{B(z) = \frac{\mu_0 I}{2} R^2 \left[\frac{1}{(R^2 + (z+\frac{s}{2})^2)^{3/2}} + \frac{1}{(R^2+(z-\frac{s}{2})^2)^{3/2}} \right]}$$

$$\frac{\partial B}{\partial z} = \frac{\mu_0 I R^2}{2} \left[\frac{-3(z+\frac{s}{2})}{(R^2+(\frac{s}{2}+z)^2)^{5/2}} - \frac{3(z-\frac{s}{2})}{(R^2+(\frac{s}{2}+z)^2)^{5/2}} \right]$$

$$\left.\frac{\partial B}{\partial z}\right|_{z=0} = \frac{\mu_0 I R^2}{2} \left[\frac{-3s/2}{(R^2+\frac{s^2}{4})^{5/2}} + \frac{3s/2}{(R^2+\frac{s^2}{4})^{5/2}} \right] = 0$$

Thus half way in between the coils $\frac{\partial B}{\partial z} = 0$ regardless of their spacing s

(b) Helmholtz configuration: For appropriate choices of s $\frac{\partial^2 B}{\partial z^2} = 0$

$$\frac{\partial^2 B}{\partial z^2} = \frac{\mu_0 I R^2}{2} \left[\frac{15(z+\frac{s}{2})^2}{(R^2+(\frac{s}{2}+z)^2)^{7/2}} + \frac{15(z-\frac{s}{2})^2}{(R^2+(z-\frac{s}{2})^2)^{7/2}} \right.$$

$$\left. - \frac{3}{(R^2+(z+\frac{s}{2})^2)^{5/2}} - \frac{3}{(R^2+(z-\frac{s}{2})^2)^{5/2}} \right]$$

Therefore

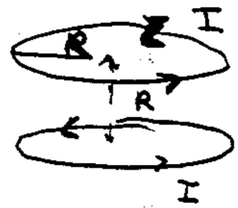
$$\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} = \frac{3\mu_0 I R^2}{2}\left[\frac{5s^2}{2\left(R^2+\frac{s^2}{4}\right)^{7/2}} - \frac{2}{\left(R^2+\frac{s^2}{4}\right)^{5/2}}\right]$$

Helmholtz configuration

$$\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} = 0 \Rightarrow \frac{5s^2}{2\left(R^2+\frac{s^2}{4}\right)^{7/2}} = \frac{2}{\left(R^2+\frac{s^2}{4}\right)^{5/2}}$$

$$\Rightarrow 5s^2 = 4\left(R^2+\frac{s^2}{4}\right) \Rightarrow s^2 = R^2$$

Thus $\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} \Rightarrow s = R$



Helmholtz coils $s = R$ $\Rightarrow$ Uniform field in center

$$\therefore\ B_{\text{Helm}}(z) = \frac{\mu_0 I R^2}{2}\left[\frac{1}{\left(\frac{5R^2}{4} - Rz + z^2\right)^{3/2}} + \frac{1}{\left(\frac{5R^2}{4} + Rz + z^2\right)^{3/2}}\right]$$

$$B(0) = \frac{\mu_0 I R^2}{\cancel{2}}\left[\cancel{2}\left(\frac{4}{5R^2}\right)^{3/2}\right] = \boxed{\frac{8\mu_0 I}{5\sqrt{5}\,R}}$$ Field at center of Helmholtz coils

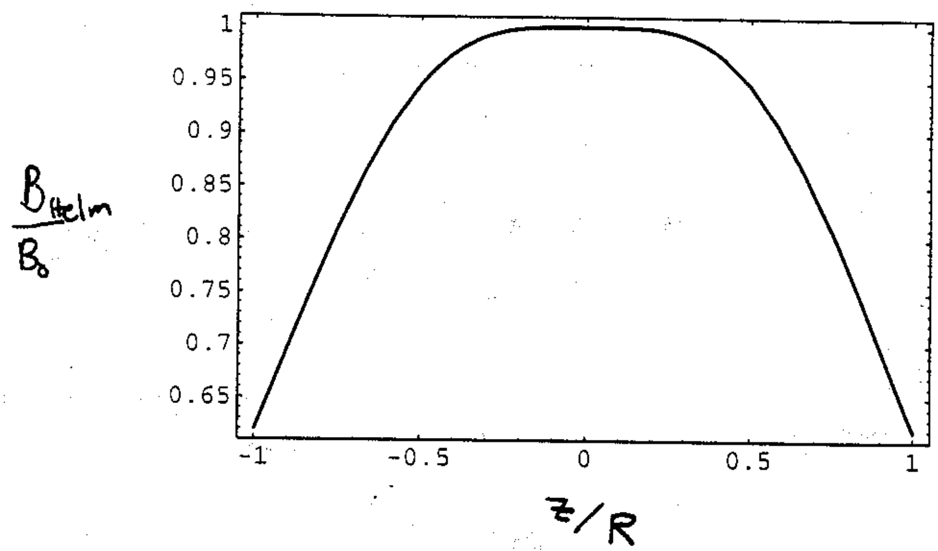
Helmholtz coils are very common place in the laboratory. For example one may want to remove the earth's magnetic field in a sensitive experiment

If $R = 10\text{ cm} = .1\text{ m}$, $B_{\text{earth}} = 0.5$ gaus $= 5\times10^{-5}$ T

$$I = \frac{BR}{8\mu_0}5\sqrt{5} = \boxed{5.56\text{ A}}$$ Requires large current

Plot of $B_{Helm}(z)$ in units of $B_0 = \frac{8\mu_0 I}{5\sqrt{5}R}$, $\bar{z} = \frac{z}{R}$

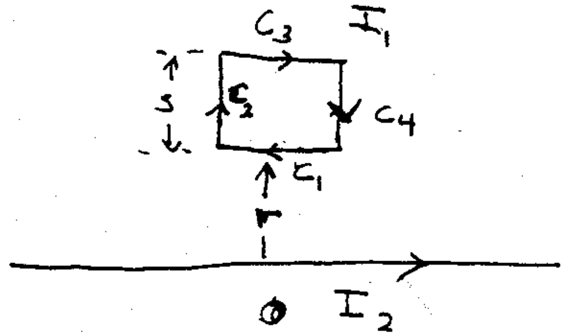
$$\frac{B_{Helm}}{B_0} = \frac{5\sqrt{5}}{16}\left[\frac{1}{\left(\frac{5}{4} - \bar{z} + \bar{z}^2\right)^{3/2}} + \frac{1}{\left(\frac{5}{4} + \bar{z} + \bar{z}^2\right)^{3/2}}\right]$$

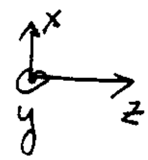


Note that the field is <u>very</u> uniform in the center.

In fact $\left.\frac{\partial^3 B}{\partial z^3}\right|_{z=0} = 0$

Problem 4: Force on a current loop





(a) The Lorentz force on a point charge ~~moving~~ is

$$\vec{F} = q(\vec{v} \times \vec{B})$$

Thus the force of a moving line charge (i.e. current carrying wire loop) is using $q\vec{v} \Rightarrow I d\vec{l}$

$$\vec{F} = \oint I d\vec{l} \times \vec{B}$$

Now, we know that the magnetic field a distance d from the infinite wire is

$$\vec{B} = \frac{\mu_0 I_2}{2\pi d}\hat{\phi}$$ where $\hat{\phi}$ is the azimulthal direction relative to the direction of I

With the x-y-z coordinates as shown,

$$d = x \, , \quad \hat{\phi} = \hat{y} \text{ (in the plane of the loop)}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I_2}{2\pi x}\hat{y}$$ (Next Page)

Now break up the loop into the four sides as shown on the previous page

$$\Rightarrow \quad \vec{F} = I_1 \int_{c_1} d\vec{l}_1 \times \vec{B} + I_1 \int_{c_2} d\vec{l}_2 \times \vec{B} + I_1 \int_{c_3} d\vec{l}_3 \times \vec{B} + I_1 \int_{c_4} d\vec{l}_4 \times \vec{B}$$

Along c_1: $d\vec{l}_1 = -dz\,\hat{z}$, $\quad \vec{B} = \frac{\mu_0 I_2}{2\pi r}\hat{y}$

C_2: $d\vec{l}_2 = +dx\,\hat{x}$, $\quad \vec{B} = \frac{\mu_0 I_2}{2\pi x}\hat{y} \quad r < x < r+s$

C_3: $d\vec{l}_3 = dz\,\hat{z}$, $\quad \vec{B} = \frac{\mu_0 I_2}{2\pi(r+s)}\hat{y}$

C_4: $d\vec{l}_4 = -dx\,\hat{x}$ $\quad \vec{B} = \frac{\mu_0 I_2}{2\pi x}\hat{y} \quad r < x < r+s$

Along C_1 and C_3, $\vec{B}$ is constant

$$\Rightarrow I_1 \int_{c_1} d\vec{l}_1 \times \vec{B} = I_1 s B(r)(-\hat{z} \times \hat{y}) = \frac{\mu_0 I_1 I_2}{2\pi r} s \,(-\hat{x}$$

$$I_1 \int_{c_3} d\vec{l}_3 \times \vec{B} = I_1 s B(r+s)(\hat{z} \times \hat{y}) = \frac{\mu_0 I_1 I_2 s}{2\pi(r+s)} (\hat{x}$$

Along C_2 and C_4 $\vec{B}$ varies

$$\Rightarrow I_1 \int_{C_2} d\vec{l}_2 \times \vec{B} = \left(\frac{\mu_0 I_1 I_2}{2\pi} \int_r^{r+s} \frac{dx}{x}\right) \hat{z}$$

$$I_1 \int_{C_4} d\vec{l}_2 \times \vec{B} = \left(\frac{\mu_0 I_1 I_2}{2\pi} \int_r^{r+s} \frac{dx}{x}\right) (-\hat{z})$$

$\therefore$ Force on contour C_2 and C_4 are equal and opposite, thus cancelling

Thus the force on the loop is

$$\vec{F} = I_1 \int_{C_1} d\vec{l}_1 \times \vec{B} + I_2 \int_{C_3} d\vec{l}_2 \times \vec{B}$$

$$\boxed{\begin{gathered}\vec{F} = -\hat{r}\, \mu_0 \frac{I_1 I_2}{2\pi} \left(\frac{s}{r} - \frac{s}{r+s} \right) \\ \hat{x} = \hat{r}\end{gathered}}$$

(b) In the limit $s \ll r$

$$\frac{s}{r+s} = \frac{s}{r}\left(\frac{1}{1+\frac{s}{r}}\right) \approx \frac{s}{r}\left(1 - \frac{s}{r}\right)$$

$$\Rightarrow \vec{F} \approx -\hat{r}\, \frac{\mu_0 I_1 I_2}{2\pi} \left(\frac{s}{r} - \frac{s}{r}\left(1 - \frac{s}{r}\right)\right)$$

$$\vec{F} \cong -\hat{r}\, \frac{\mu_0 I_2 I_1 s^2}{2\pi r^2}$$

Now with $\vec{m} = I_1 s^2 \hat{\phi} = I\,(\text{Area of loop})\, \hat{n}$ ~ normal to loop

and $\vec{B}$ at loop $\vec{B} = \hat{\phi}\, \frac{\mu_0 I_2}{2\pi r}$

$$\vec{\nabla}(\vec{m} \cdot \vec{B}) = \vec{\nabla}\left((I_1 s^2 \hat{\phi}) \cdot \left(\frac{\mu_0 I_2}{2\pi r} \hat{\phi} \right)\right)$$

$$= \vec{\nabla}\left(\frac{\mu_0 I_1 I_2 s^2}{2\pi r} \right) = \frac{d}{dr}\left(\frac{\mu_0 I_1 I_2 s^2}{2\pi r} \right) \hat{r}$$

$$\Rightarrow \vec{\nabla}(\vec{m} \cdot \vec{B}) = -\hat{r}\, \frac{\mu_0 I_1 I_2 s^2}{2\pi r^2} = \text{Force of loop}$$

This result is general!