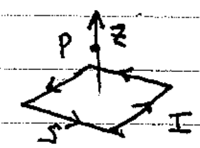
Physics 405

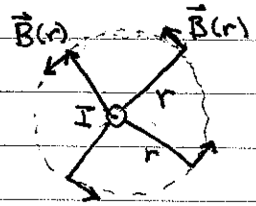
P.S #11 Solutions

Problem 1 Find $\vec{B}$ along the axis of a square current loop



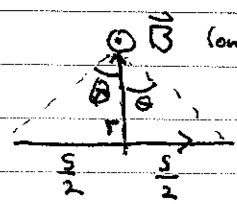
Use the superposition of the $\vec{B}$ field for each side of the square.

Current out of page



$$\vec{B}(r) = \frac{\mu_0 I}{4\pi r}\left(\sin\theta_2 + \sin\theta_1\right)\hat{\phi}$$

Current in page

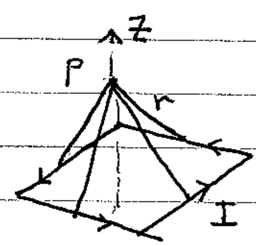


For the geometry here the point P is such that $\theta_1 = \theta_2 \equiv \theta$

where $\tan\theta = \dfrac{s}{2r}$

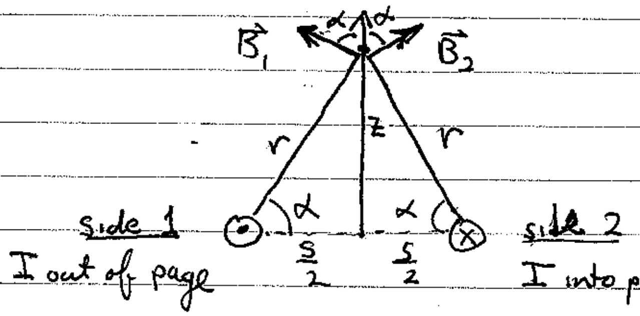
$\sin\theta = \dfrac{s}{\sqrt{4r^2+s^2}}$

Now we add the vector contributions of each side



The point P is the same distance r from each side of the squeen

Consider two sides (Cross section of loop)



We see from this geometry that the vector sum of $\vec{B}_1$ and $\vec{B}_2$ has only a z-component

$$\Rightarrow \vec{B}_1 + \vec{B}_2 = \left(\vec{B}_1(r)\cdot\hat{z} + \vec{B}_2(r)\cdot\hat{z}\right)\hat{z} = 2B(r)\cos\alpha\,\hat{z}$$

$$\cos\alpha = \frac{s}{2r} = \frac{s}{2\sqrt{z^2+\frac{s^2}{4}}}$$

The same is true when we add the contributions from sides 3 and 4

$$\Rightarrow \vec{B}_{total} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = 4B(r)\cos\alpha\,\hat{z}$$

$$= 4\left(\frac{\mu_0 I}{4\pi r}\cdot 2\sin\theta\right)\cos\alpha\,\hat{z}$$

$$= \frac{2\mu_0 I}{\pi r}\sin\theta\cos\alpha\,\hat{z} = \frac{2\mu_0 I}{\pi\sqrt{z^2+\frac{s^2}{4}}}\sin\theta\cos\alpha\,\hat{z}$$

$$\sin\theta = \frac{s}{2\sqrt{r^2+\frac{s^2}{4}}} = \frac{s}{2\sqrt{z^2+\frac{s^2}{2}}} \qquad \left(\text{since } r^2 = z^2+\frac{s^2}{4}\right)$$

$$\therefore \boxed{\vec{B}_{total}(z) = \frac{\mu_0 I s^2}{2\pi\left(z^2+\frac{s^2}{4}\right)\left(z^2+\frac{s^2}{2}\right)^{1/2}}\,\hat{z}}$$

(b) In the limit $z >> s$

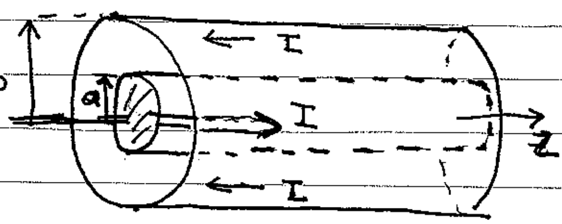
$$\vec{B}_{total}(z) \approx \frac{\mu_0 I s^2}{2\pi z^3}\hat{z} = \frac{\mu_0}{4\pi}\left(\frac{2m}{z^3}\right)\hat{z}$$

Dipole field on axis

Magnetic dipole moment $\boxed{m = I s^2}$

As for the circular loop $m = I$ (Area of loop)

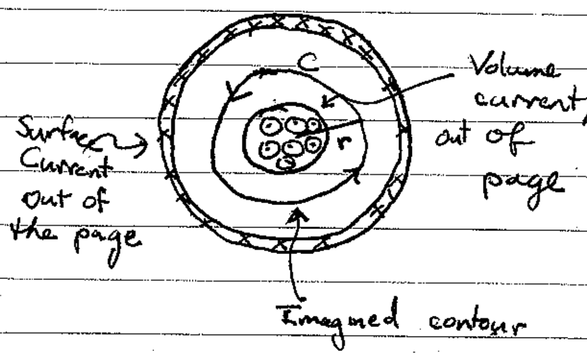
Problem 2



From the symmetry of the current distribution we know that

$$\boxed{\vec{B} = B(r)\,\hat{\phi}}$$

Use Ampere's Law: Take contour in $\hat{\phi}$ direction



Ampere's Law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\Rightarrow 2\pi r\, B(r) = \mu_0 I_{enc}$$

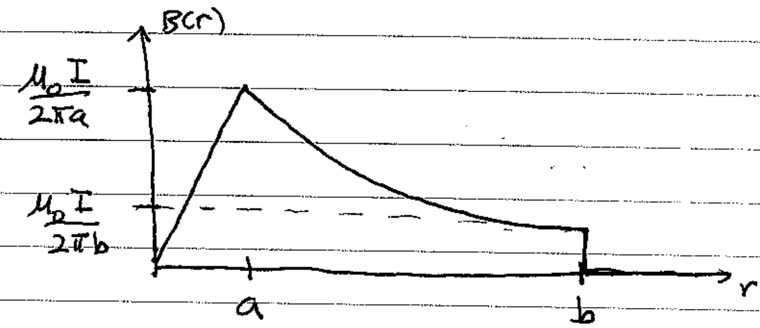
(since $|\vec{B}|$ constant over contour)

$$I_{enc} = \mu_0 \int \vec{J}\cdot\hat{n}\,dA = \begin{cases} \frac{I}{\pi a^2}\pi r^2 = I\frac{r^2}{a^2} & r \le a \\ I & a \le r < b \\ 0 & r > b \end{cases}$$

since current on outer cylinder cancels current in inner "

$$\therefore \boxed{B(r) = \begin{cases} \dfrac{\mu_0 I r}{2\pi a^2} & r \le a \\ \dfrac{\mu_0 I}{2\pi r} & a \le r < b \\ 0 & r > b \end{cases}}$$

(b) Sketch $B(r)$



$B(r)$
- linearly increases $0 < r < a$
- falls off like $\frac{1}{r}$ for $a < r < b$
- Is zero for $r > b$

(c) At $r = b$ we see that there is a discontinuity in $B(r)$

$$B(r = b_-) = \frac{\mu_0 I}{2\pi a}, \quad B(r = b_+) = 0$$

This is because there is surface current at $r = b$

$$K = \frac{\text{Current}}{\text{length}} = \frac{I}{2\pi b}$$

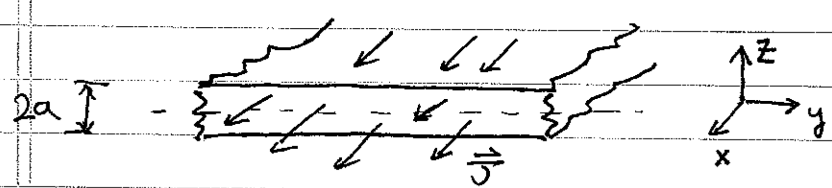
We know that the component of $\vec{B}$ parallel to the surface suffers a discontinuity

$$\Delta B_{\parallel}\Big|_{\text{surface}} = \mu_0 K$$

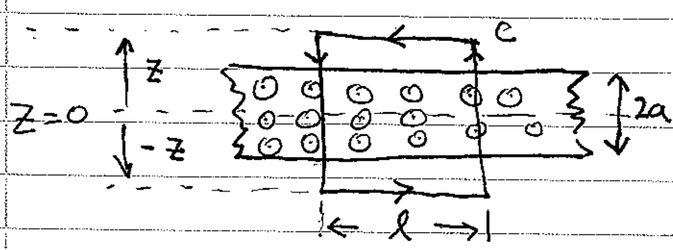
Here the parallel component is in the $\hat{\phi}$ direction

$$\Delta B_{\parallel}\Big|_{\text{surface}} = B_\phi(r = b_-) - B_\phi(r = b_+) = \frac{\mu_0 I}{2\pi a} = \mu_0 K$$

Problem 3: Slab of current with uniform current density $\vec{J} = J\hat{x}$



This current distribution has the same symmetry as ~~the~~ a plane of surface current (in x-y plane) flowing in the x-direction which we treated in class. There we argued that the direction of $\vec{B}$ must be $\hat{y}$. In addition since $\vec{J}$ is uniform in x and y $\vec{B}$ can only depend on z $\Rightarrow$ $\vec{B} = B(z)\,\hat{y}$. Now we can use Ampère's Law by choosing contours along $\hat{y}$ (or $\perp$ to $\hat{y}$)



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$= -B(z)l + 0 + B(-z)l + 0$$

$$= \mu_0 I_{enc}$$

(Note - sign for top leg)

Now from symmetry and right-hand-rule,

$$B(-z) = -B(z)$$

$$\Rightarrow \quad -2lB(z) = \mu_0 I_{enc} \quad \Rightarrow \quad B(z) = -\frac{\mu_0 I_{enc}}{2l} \quad z > 0$$

$$B(z) = +\frac{\mu_0 I_{enc}}{2l} \quad z < 0$$

Now we must find I_{enc} for different values of z

$$I_{enc} = \int_S \vec{J} \cdot \hat{n}\, dA$$

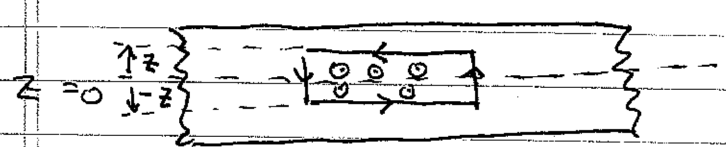
where S is the surface bounded by the contour C, and $\hat{n}$ is the normal to S in the direction determined by the right hand rule associate with the direction of C.

For the contour drawn on the previous page
$\hat{n} = \hat{x}$ (check R.H.R.)

$\Rightarrow I_{enc} = J\,(\text{Area of } S) = J\,(2a)\,\ell$ (for $|z| > a$)

$$\Rightarrow \vec{B}(z) = \begin{cases} -\dfrac{\mu_0 I_{enc}}{2\ell}\,\hat{y} = -\mu_0 J a\,\hat{y} & z > a \\ +\dfrac{\mu_0 I_{enc}}{2\ell}\,\hat{y} = +\mu_0 J a\,\hat{y} & z < -a \end{cases}$$

What about inside the slab?

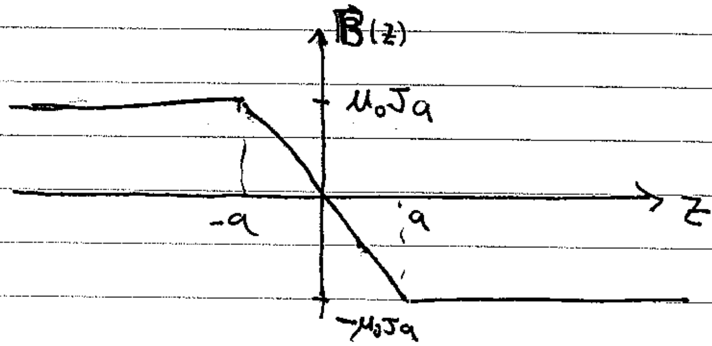


Now for $|z| < a$
$I_{enc} = J\,(\text{Area of } S)$
$= J\,(2z)\,\ell$

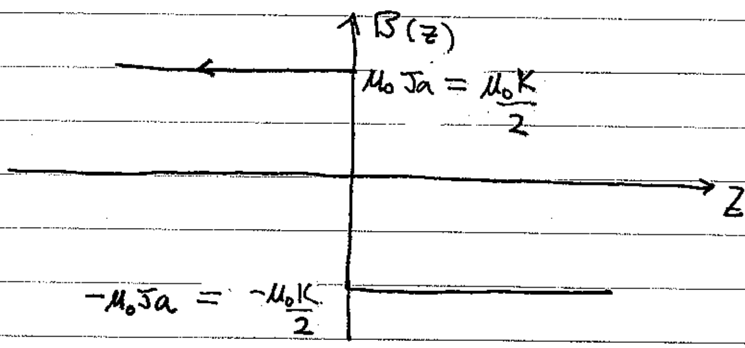
$$\Rightarrow B(z) = \frac{-\mu_0 I_{enc}}{2\ell} = -\mu_0 J z \qquad |z| < a$$

Thus the final solution is

$$\boxed{\vec{B} = \begin{cases} -\mu_0 J a \,\hat{y} & z \geq a \\ -\mu_0 J z \,\hat{y} & -a \leq z \leq a \\ \mu_0 J a \,\hat{y} & z \leq -a \end{cases}}$$



In the limit $J(2a) \to K$ as $a \to 0$



Recover solution for the plane of current

B is discontinous at $z = 0$ by an amount $\mu_0 K$

This is expected from our solution for the surface current