Physics 405

Problem set #12 Solutions

Problem #1

Maxwell's equations for electro/magneto statics with magnetic monopoles

$$\vec{\nabla}\cdot\vec{E} = \rho_E/\epsilon_0 \qquad \vec{\nabla}\times\vec{E} = \frac{\vec{J}_M}{\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{B} = \mu_0\rho_M \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}_E$$

(a) Show that solution is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int dV'\,\rho_E(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{1}{4\pi\epsilon_0}\int dV'\,\vec{J}_M(\vec{r}\,')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int dV'\,\rho_M(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{\mu_0}{4\pi}\int dV'\,\vec{J}_E(\vec{r}\,')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

We must take 4 derivatives:

$$\vec{\nabla}\cdot\int dV'\,\rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}, \qquad \vec{\nabla}\cdot\int dV'\,\vec{J}(\vec{r}\,')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec{\nabla}\times\int dV'\,\rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}, \qquad \vec{\nabla}\times\int dV'\,\vec{J}(\vec{r}\,')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

where $\rho = \rho_E$ or ρ_M, $\vec{J} = \vec{J}_E$ or $\vec{J}_M$

and $\vec{\nabla}$ is the "del operator" with respect to "unprimed" coordinates. Also, $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

(1) $$\vec{\nabla}\cdot\int dV'\,\rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \int dV'\,\vec{\nabla}\cdot\left(\rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$= \int dV'\,\rho(\vec{r}\,')\,\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$ (since $\vec{\nabla}$ acts only on "unprimed variables")

Now $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\vec{\nabla}\frac{1}{\mathscr{r}} \Rightarrow \vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\nabla^2\frac{1}{\mathscr{r}} = 4\pi\delta^{(3)}(\vec{\mathscr{r}})$
$= 4\pi\delta^{(3)}(\vec{r}-\vec{r}\,')$

$$\therefore \vec{\nabla}\cdot\int dV'\,\rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\int dV'\,\rho(\vec{r}\,')\,\delta^{(3)}(\vec{r}-\vec{r}\,') = 4\pi\rho(\vec{r})$$

Problem 2

(a) If $\vec{B}$ is uniform show that $\vec{\nabla}\times\vec{A} = \vec{B}$, $\vec{\nabla}\cdot\vec{A} = 0$ if we chose $\vec{A} = -\frac{\vec{r}\times\vec{B}}{2}$

Use the "product rules" on the inside cover of Griffiths

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{2}\vec{\nabla}\cdot(\vec{r}\times\vec{B}) = -\frac{\vec{B}}{2}\cdot(\vec{\nabla}\times\vec{r}) + \frac{\vec{r}}{2}\cdot\underbrace{(\vec{\nabla}\times\vec{B})}_{0} \quad (\vec{B}\text{ uniform})$$

$$\vec{\nabla}\times\vec{r} = 0 \qquad \text{proof:} \quad \vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = 0 \ \checkmark$$

$$\vec{\nabla}\times\vec{A} = -\frac{1}{2}\vec{\nabla}\times(\vec{r}\times\vec{B}) = -\frac{1}{2}\left\{(\vec{B}\cdot\vec{\nabla})\vec{r} - (\vec{r}\cdot\vec{\nabla})\vec{B} + \vec{r}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{r})\right\}$$

Now $\vec{\nabla}\cdot\vec{B} = 0$ and $(\vec{r}\cdot\vec{\nabla})\vec{B} = 0$ (since $\vec{B}$ is uniform)

$$\Rightarrow \vec{\nabla}\times\vec{A} = -\frac{1}{2}\left\{(\vec{B}\cdot\vec{\nabla})\vec{r} - \vec{B}(\vec{\nabla}\cdot\vec{r})\right\}$$

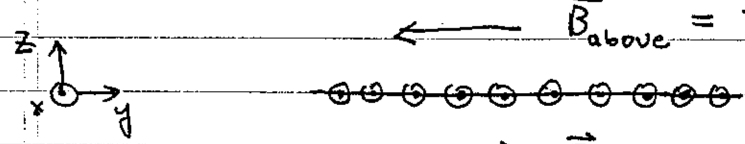
Aside $(\vec{B}\cdot\vec{\nabla})\vec{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$

$$= B_x\hat{x} + B_y\hat{y} + B_z\hat{z} = \vec{B}$$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = -\frac{1}{2}\left\{\vec{B} - 3\vec{B}\right\} = \vec{B} \ \checkmark$$

(b) For the uniform sheet of current we found



$$\vec{B}_{above} = -\frac{\mu_0 K}{2}\hat{y}$$

$$\vec{K} = K\hat{x} \quad \text{(out of page)}$$

$$\vec{B}_{below} = \frac{\mu_0 K}{2}\hat{y}$$

$$\Rightarrow \vec{A}_{above} = -\frac{1}{2}\left(\vec{r}\times\vec{B}_{above}\right) = -\frac{1}{2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ 0 & -\frac{\mu_0 K}{2} & 0 \end{vmatrix}$$

$$\Rightarrow \vec{A}_{above} = +\frac{\mu_0 K}{4}\left(x\hat{z} - z\hat{x}\right)$$

Similary $\vec{A}_{below} = -\frac{1}{2}\left(\vec{r}\times\vec{B}_{below}\right) = -\vec{A}_{above} = -\frac{\mu_0 K}{4}\left(x\hat{z} - z\hat{x}\right)$

$$\Rightarrow \boxed{\vec{A} = \text{sign}(z)\,\frac{\mu_0 K}{4}\left(x\hat{z} - z\hat{x}\right)}$$

This form of the vector potential, though correct, is strange. Note that $\vec{A}$ depends on the x-coordinate of observation. But the current sheet is infinite in x, so how can we know where $x=0$ is!? Of course $\vec{B}$ is independent of x, so all is OK. Another choice for $\vec{A}$ which doesn't have this unphysical character is

$$\vec{A}' = -\text{sign}(z)\,\frac{\mu_0 K}{2}\,z\,\hat{x}$$

Then you can easily convince yourself that

$$\vec{\nabla}\cdot\vec{A}' = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{A}' = \vec{B}$$

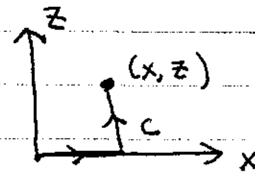
The two choices of $\vec{A}$ must be related by a gauge transformation. Let's see this explicitly for $z > 0$

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \quad \text{for some scalar } \chi. \text{ Find } \chi$$

$$\vec{A}' - \vec{A} = -\frac{\mu_0 K}{2} z\hat{x} - \frac{\mu_0 K}{4}(x\hat{z} - z\hat{x})$$

$$= -\frac{\mu_0 K}{4}(x\hat{z} + z\hat{x}) = \vec{\nabla}\chi = \frac{\partial\chi}{\partial x}\hat{x} + \frac{\partial\chi}{\partial y}\hat{y} + \frac{\partial\chi}{\partial z}$$

Integrating along the contour



we find

$$\chi = -\frac{\mu_0 K}{4} xz \Rightarrow \vec{\nabla}\chi = -\frac{\mu K}{4}(x\hat{z} + z\hat{x}) \checkmark$$

Moral of the story

Even by restricting $\vec{\nabla}\cdot\vec{A} = 0$, there are still an infinite number of vector potentials which generate the same magnetic field $\vec{B}$, which are related by a gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

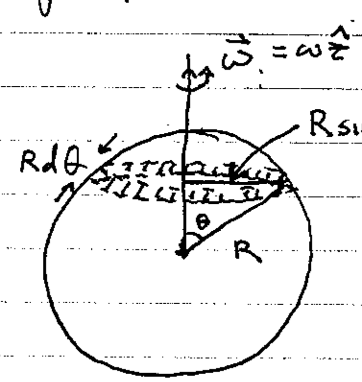
$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}' = 0 \quad \text{if} \quad \nabla^2\chi = 0$$

Problem 3: Sphere with uniform surface charge σ rotating with ~~angl~~ angular frequency ω

From P.S. #10, we found the surface current

$$\vec{K}(\theta) = \sigma\omega R \sin\theta \ \hat{\phi}$$

(a) Breaking up the surface into current rings



The ring at (R,θ) has

- Radius of ring $= R\sin\theta$
- differential current in ring $dI(\theta) = K\,R d\theta = \sigma\omega R^2 \sin\theta\, d\theta$

The magnetic dipole moment associated with this differential ring of current is

$$d\vec{m} = dI(\theta)\,(\text{Area of ring})\,\hat{z} = dI(\theta)\ \pi (R\sin\theta)^2\,\hat{z}$$
$$= \pi\sigma\omega R^4 \sin^3\theta\,\hat{z}\ d\theta$$

Thus the total magnetic dipole moment is

$$\vec{m} = \int d\vec{m} = \int_0^{\pi} d\theta\ \pi\sigma\omega R^4 \sin^3\theta\ \hat{z}$$
$$= \pi\sigma\omega R^4 \int_0^{\pi} d\theta\,(1-\cos^2\theta)\sin\theta\ d\theta\ \hat{z}$$
$$= \pi\sigma\omega R^4 \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi} \hat{z}$$
$$= \pi\sigma\omega R^4 \left[+1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right]\ \hat{z}$$

$$\Rightarrow \boxed{\vec{m} = \frac{4\pi R^3}{3}(\sigma\omega R)\ \hat{z}}$$

As in problem set #10

(b) The vector potential at point $\vec{r}$ with $|\vec{r}| >> R$ is

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi}$$

and the magnetic field is

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \left(\frac{2m\cos\theta}{r^3} \hat{r} + \frac{m \sin\theta}{r^3} \hat{\theta} \right)$$

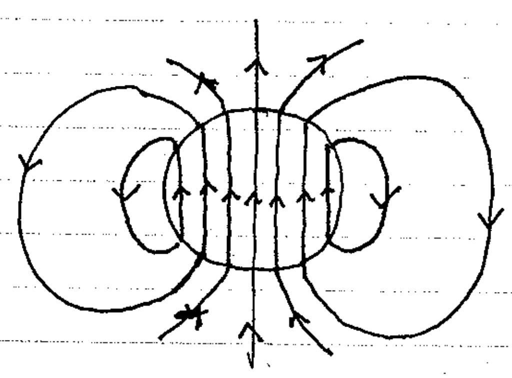
$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 \sigma \omega R^4}{3} \frac{1}{r^3} \left(2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right)}$$

Note at $\theta = 0$ $\quad r = z \quad \hat{r} = \hat{z}$

$\Rightarrow$ On axis: $\quad \vec{B}(z) = \dfrac{2\mu_0 \sigma \omega R^4}{3 z^3} \hat{z}$

This is the answer we found in P.S. #10 using the Biot-Savart law for any $z \geq R$.

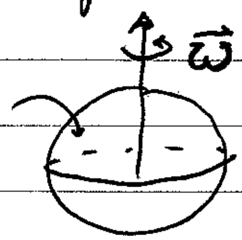
The answer boxed above is exact for any $r > R$. The magnetic field associated with a rotating sphere of charge is a pure dipole field outside the sphere. Inside the field is uniform $\vec{B}_{inside} = \frac{2}{3} \mu_0 \sigma \omega R \hat{z}$



Note $\vec{\nabla} \cdot \vec{B} = 0$

The discontinuity in of the tangential component of $\vec{B}$ follows from the boundary conditions

(c) Gyro-magnetic ratio

$\sigma = \frac{Q}{4\pi R^2}$ 

Spinning ball, mass M

$$\vec{L} = I_{ball}\,\vec{\omega}$$

moment of interia (spherical shell)

Mass on shell, $I_{ball} = \frac{2}{3} M R^2$

$$\Rightarrow \vec{L} = \frac{2}{3} M R^2 \vec{\omega}$$

Magnetic moment: $\vec{m} = \frac{4\pi R^3}{3}\left(\frac{Q}{4\pi R^2}\,\omega R\right)\hat{z}$

$$\vec{m} = \frac{\pi R^2}{3} Q\,\vec{\omega}$$

$$\Rightarrow \boxed{\frac{|\vec{m}|}{|\vec{L}|} = \frac{Q}{2M}}$$ Gyromagnetic ratio