Physics 405

Problem Set #12: DUE 5/4/2007

Read Griffiths Chap. 6

Problem 1

A World of Magnetic Monopoles

In our universe, as far as we know, there are no magnetic monopoles (magnetic charges). In this case electric fields are generated by electric charges, and magnetic fields by electric current (moving electric charges). For steady currents (electro- and magneto-static), Maxwell's equations read

$$\nabla \cdot \mathbf{E} = \frac{\rho_E}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_E ,$$

where the subscript *E* is shown to emphasize that these are do to electric charges, with solutions

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\ \rho_E(\mathbf{r}')\frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}_E(\mathbf{r}')\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} .$$

In addition, we can define a scalar potential V_E, and a vector potential $\mathbf{A}_E$, such that $\mathbf{E} = -\nabla V_E$, and $\mathbf{B} = \nabla \times \mathbf{A}_E$ (with $\nabla \cdot \mathbf{A} = 0$), whose solutions are

$$V_E(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dV'\ \frac{\rho_E(\mathbf{r}')}{|\boldsymbol{\mathscr{r}}|} , \quad \mathbf{A}_E(\mathbf{r}) = \frac{\mu_0}{4\pi}\int dV'\ \frac{\mathbf{J}_E(\mathbf{r}')}{|\boldsymbol{\mathscr{r}}|} ,$$

where $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$.

Now suppose there **were** magnetic monopoles. Magnetic charges would then be a source for the magnetic field, and moving magnetic monopoles would be a source for electric fields (as electric currents are sources magnetic fields). The modified Maxwell equation for electro- and magneto-statics may then appear as:

$$\nabla \cdot \mathbf{E} = \frac{\rho_E}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = \mu_0 \rho_M$$

$$\nabla \times \mathbf{E} = \frac{\mathbf{J}_M}{\varepsilon_0} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_E$$

where $(\rho_E, \mathbf{J}_E)$ are the charge and current density for electric monopoles, and $(\rho_M, \mathbf{J}_M)$ for the magnetic monopoles.

(a) Show that in this universe

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \; \rho_E(\mathbf{r}') \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} \quad + \frac{1}{4\pi\varepsilon_0} \int d^3r' \; \mathbf{J}_M(\mathbf{r}') \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \rho_M(\mathbf{r}') \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} \quad + \frac{\mu_0}{4\pi} \int d^3r' \; \mathbf{J}_E(\mathbf{r}') \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}$$

(That is show that these satisfy the new "Maxwell's equations.
Hint: follow Griffiths sect. 5.3.2).

(b) Define a pair of scalar and vector potentials $(V_E, \mathbf{A}_E)$, and $(V_M, \mathbf{A}_M)$ such that

$$\mathbf{E} = -\nabla V_E + \nabla \times \mathbf{A}_M, \qquad \mathbf{B} = -\nabla V_M + \nabla \times \mathbf{A}_E$$

Problem 2

(a) Show that in a region of space where **B** is *uniform*, the vector potential is $\mathbf{A}(\mathbf{r}) = -(\mathbf{r}\times\mathbf{B})/2$. That is show $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$

(b) Using the form above, find the vector potential for an infinite plane (x-y plane) of surface current, with $\mathbf{K} = K\hat{x}$.

(c) Show that under a gauge transformation the vector potential takes the form,
$\mathbf{A} = -sign(z)\frac{\mu_0 K}{2} z \, \hat{x}$.

Problem 3

(a) Find the magnetic dipole moment vector of a sphere of radius R uniformly covered with surface charge σ rotating about the z-axis with angular velocity ω.

(b) What is the magnetic field for distance $r \gg R$?

(The answer you find is actually **exact** for any $r > R$. A rotating sphere of charge gives rise to a *pure* dipole field as discussed in class).

(c) If the mass *m* of the sphere is concentrated at the radius (i. e. a massive spherical shell of radius R) and the total charge on the sphere is *q*, what is the "gyromagnetic ratio" : the ratio of the magnetic moment to the angular momentum.

(The permanent magnetic moment of the electron due to its "spin" is a factor of two different from this model: a classical spinning sphere of charge)