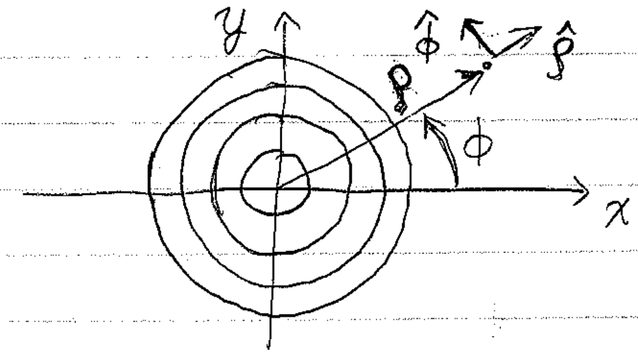
UNM Physics 405: Lecture 2

I. Curvilinear Coordinates

The "rectangular grid" of Cartesian coordinates is not a "natural symmetry"

Consider Polar Coordinates (ρ, ϕ)



Useful when there is <u>rotational</u> symmetry about an axis

Position vector in the plane:

$$\vec{\rho} = x\,\hat{x} + y\,\hat{y} = \rho\,\hat{\rho}$$

Relationships between polar and Cartesian coords:

$$\rho = |\vec{\rho}| = \sqrt{x^2+y^2}, \quad x = \rho\cos\phi$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right), \quad y = \rho\sin\phi$$

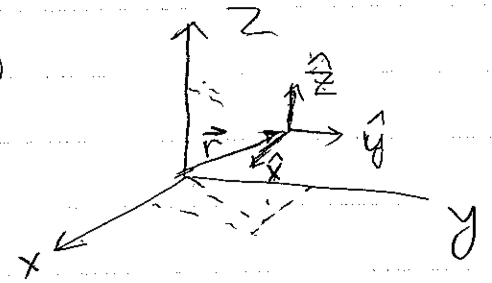
$$\Rightarrow \hat{\rho} = \frac{\vec{\rho}}{|\vec{\rho}|} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = \hat{z}\times\hat{\rho} = \cos\phi\,\hat{y} - \sin\phi\,\hat{x} = \frac{x\hat{y} - y\hat{x}}{\sqrt{x^2+y^2}}$$

B-field of wire $\Rightarrow$ $\boxed{\vec{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{2I_0}{\rho}\hat{\phi}}$

Curvilinear Coordinates in 3D

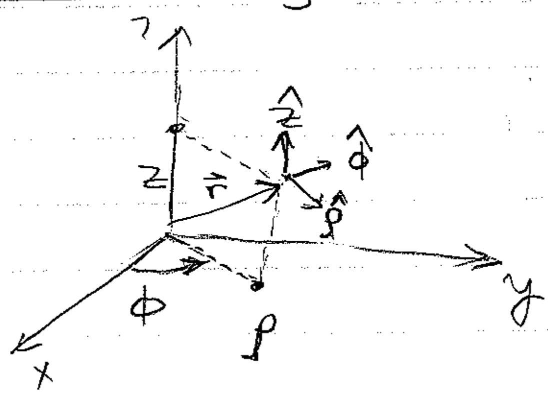
Cartesian (x, y, z)

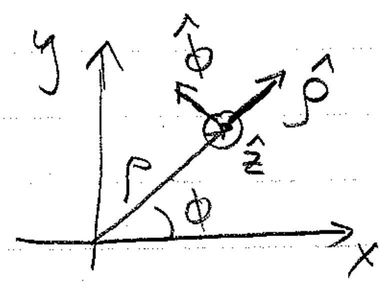


$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

Note: $\hat{x}, \hat{y}, \hat{z}$ are the same vector @ any $\vec{r}$

Cylindrical (ρ, ϕ, z)





$$\rho = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z$$

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

$$\hat{\rho} = \frac{\vec{\rho}}{|\vec{\rho}|} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

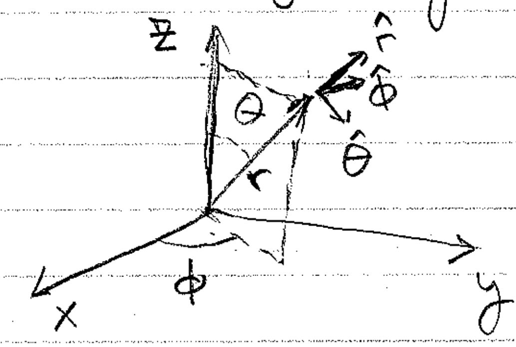
$$\hat{z}$$

Unit vectors depend on position!

Spherical Coords (r, θ, ϕ) : Most important!

Natural Symmetry – rotation in any direction

$\vec{r} = r\hat{r}$



$0 \le r < \infty$

$0 \le \theta \le \pi$

$0 \le \theta \le 2\pi$

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = \rho\cos\phi = r\sin\theta\cos\phi$$
$$y = \rho\sin\phi = r\sin\theta\sin\phi \quad , \quad z = r\cos\theta$$

$$\boxed{\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}}$$

$$\hat{\theta} = \hat{\phi} \times \hat{r} = ?$$

$$\boxed{\hat{\phi} = \cos\phi\,\hat{y} - \sin\phi\,\hat{x}}$$

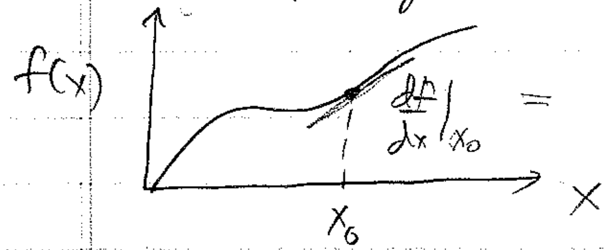
$$\hat{\theta} = (\cos\phi\,\hat{y} - \sin\phi\,\hat{x}) \times (\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})$$

$$= \sin\theta\cos^2\phi\,(-\hat{z}) - \sin\theta\sin^2\theta\,(\hat{z}) + \cos\theta\cos\phi\,(\hat{x})$$
$$- \cos\theta\sin\phi\,(-\hat{y})$$

$$\Rightarrow \boxed{\hat{\theta} = \cos\theta(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) - \sin\theta\,\hat{z}}$$

II. Derivatives of Fields

Recall, single variable calculus:



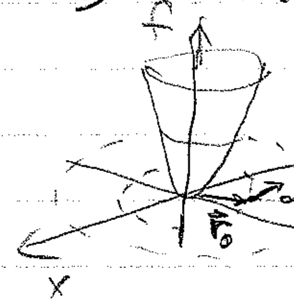
$$\left.\frac{df}{dx}\right|_{x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \text{local slope of curve}$$

Linear approx $\quad \Delta f = \left(\left.\frac{df}{dx}\right|_{x_0}\right) \Delta x \quad$ (for small Δx)

Consider now scalar field $f(\vec{r})$

How fast does f change at $\vec{r}_0$ as $\vec{r}_0 \Rightarrow \vec{r}_0 + \Delta\vec{s}$

Example: 2D $\quad f(x,y) \equiv A(x^2+y^2)$



← Paraboloid

$d\vec{s} = dx\,\hat{x} + dy\,\hat{y}$ (an arbitrary displacement from $\vec{r}_0$)

"Directional derivative:

$$\left.\frac{df}{ds}\right|_{\vec{r}_0} = \lim_{\Delta s \to 0} \left(\frac{f(x_0+\Delta x,\, y_0+\Delta y) - f(x_0,y_0)}{\Delta s} \right)$$

$$= \left.\frac{\partial f}{\partial x}\right|_{x_0} \frac{dx}{ds} + \left.\frac{\partial f}{\partial y}\right|_{y_0} \frac{dy}{ds} \qquad \text{where } ds = |d\vec{s}|$$

$$= \left(\left.\frac{\partial f}{\partial x}\right|_{x_0} \hat{x} + \left.\frac{\partial f}{\partial y}\right|_{y_0} \hat{y} \right) \cdot \left(\frac{dx}{ds}\hat{x} + \frac{dy}{ds}\hat{y} \right)$$

$$\Rightarrow \left.\frac{df}{ds}\right|_{\vec{r}_0} = \underbrace{\left.\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}\right)\right|_{\vec{r}_0}}_{\left.\overrightarrow{\text{grad}}\, f\right|_{\vec{r}_0}} \cdot \underbrace{\left(\frac{d\vec{s}}{ds}\right)}_{\hat{n}}$$

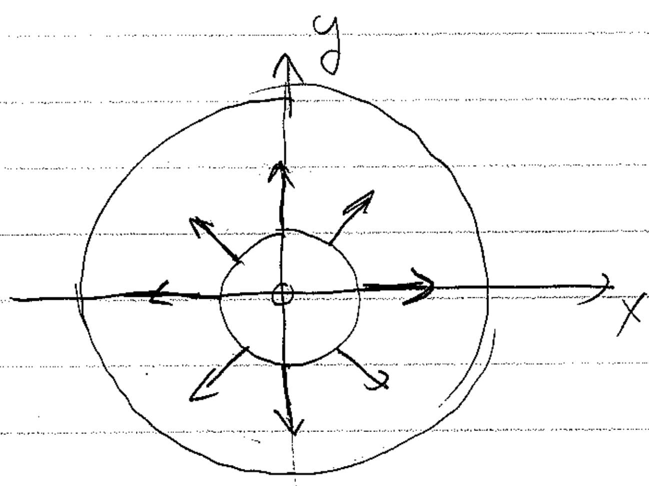
$\hat{n}$ (Unit vector in director we look @ change in f)

$$\Rightarrow \left.\frac{df}{ds}\right|_{\vec{r}_0} = |\overrightarrow{\text{grad}}\, f| \cos\Theta$$

Θ ← angle with $\hat{n}$

⇒ Gradient of f points along direction a maximum change in f

⇒ $\overrightarrow{\text{grad}}\, f$ points along direction perpendicular to the contours of f



$f(x,y) = A(x^2+y^2)$

⇒ gradient

$\overrightarrow{\text{grad}} f = 2A(x\hat{x} + y\hat{y})$

$= 2A\vec{\rho}$

$= 2A\rho\hat{\rho}$

The differential operator $\vec{\nabla} \equiv$ "del"

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \qquad (\text{in } 3D)$$

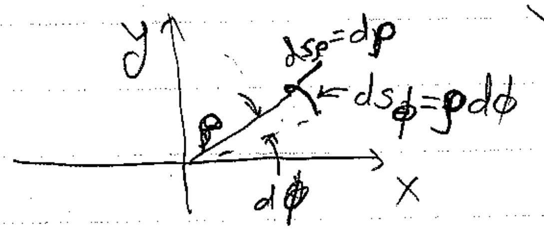
$$\Rightarrow \overrightarrow{\text{grad}}\, f = \vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

* Important note: Not same form in curvilinear coords because unit vector depend on position

• $\vec{\nabla} f$ in Cylindrical Coords

Need to consider: $d\vec{s} = ds_\rho\,\hat{\rho} + ds_\phi\,\hat{\phi} + ds_z\,\hat{z}$

$$= d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}$$



Now: $df = \frac{df}{d\rho} d\rho + \frac{df}{d\phi} d\phi + \frac{df}{dz} dz$

$$= \frac{df}{d\rho} ds_\rho + \left(\frac{df}{d\phi}\frac{1}{\rho}\right) ds_\phi + \left(\frac{df}{dz}\right) ds_z$$

$$= (\vec{\nabla} f) \cdot d\vec{s}$$

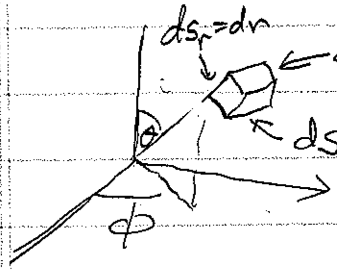
$$\Rightarrow \vec{\nabla} f(\rho, \phi, z) = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\boxed{\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}}$$

Check: Our example: $f(x,y) = A(x^2+y^2) = A\rho^2$

$$\Rightarrow \vec{\nabla} f = \hat{\rho}\frac{\partial f}{\partial \rho} = \hat{\rho}\,(2A\rho) \checkmark$$

$\vec{\nabla} f$ in spherical coords



$ds_r = dr$

$ds_\theta = r\,d\theta$

$ds_\phi = \rho\, d\phi = r\sin\theta\, d\phi$

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi = (\vec{\nabla} f)\cdot d\vec{s}$$

$$= \frac{\partial f}{\partial r}ds_r + \frac{1}{r}\frac{\partial f}{\partial \theta}ds_\phi + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}ds_\theta$$

$$\Rightarrow \boxed{\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}}$$

Example: $f = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$

$$\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} = \frac{1}{4\pi\epsilon_0}\, q\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r}$$

$$= \frac{1}{4\pi\epsilon_0}\, q\left(-\frac{1}{r^2}\right)\hat{r}$$

$= -\vec{E}$ of point charge!