UNM Physics 405: Lecture 3
Differential Vector Calculus

- So far we have seen the gradient of <u>scalar field</u> $f(\vec{r})$

$$\vec{\nabla} f = \text{vector field} \perp \text{to surfaces of const. } f$$

- <u>Derivatives of vectors fields</u> $\vec{V}(\vec{r})$

Two basic derivatives: $\begin{cases} \text{Divergence} = \vec{\nabla}\cdot\vec{V} \\ \text{Curl} = \vec{\nabla}\times\vec{V} \end{cases}$

In <u>Cartesian Coords</u>:

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}V_x + \hat{y}V_y + \hat{z}V_z\right)$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}}$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \end{vmatrix}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{V} = \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)}$$

Geometrical meaning

- Divergence @ $\vec{r} \Rightarrow$ Field is locally "diverging" from or towards the point $\vec{r}$
- Curl @ $\vec{r} \Rightarrow$ Field is locally "curling" around point $\vec{r}$ (direction according to R.H.R.)

These interpretations follow from a more formal definition of Div and Curl (next lecture)

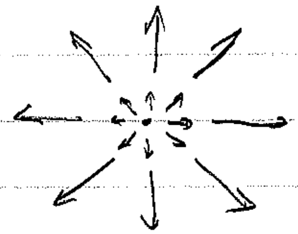
Examples

Let $\vec{V}_1(\vec{r}) = A\vec{r}$, $\vec{V}_2(\vec{r}) = A(y\hat{x} - x\hat{y})$

$$\Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{V}_1 = A\,\vec{\nabla}\cdot\vec{r} = A\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3A \\ \vec{\nabla}\times\vec{V}_1 = A(\vec{\nabla}\times\vec{r}) = 0 \end{cases}$$

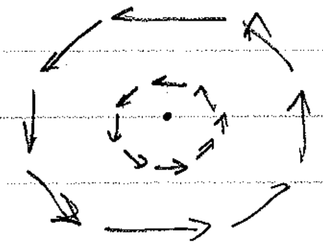
$$\Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{V}_2 = 0 \\ \vec{\nabla}\times\vec{V}_2 = A \quad \hat{z}\left(\frac{\partial V_{2y}}{\partial x} - \frac{\partial V_{2x}}{\partial y}\right) \\ \qquad = A\,\hat{z}\left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y}\right) = 2A \end{cases}$$

Geometrically



$\vec{V}_1(\vec{r})$

Divergence @ origin



$\vec{V}_2(\vec{r})$

Curl @ origine

Note: the formula's for div and curl are much more complicated in curvilinear coordinates since, e.g., $\hat{r}, \hat{\theta}, \hat{\phi}$ vary as as function of $\vec{r}$. See Griffiths inside cover

E.g.: In Spherical Coords:

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta V_\phi) - \frac{\partial V_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial}{\partial r}(rV_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right]\hat{\phi}$$

Yuk!

Product Rules and Second Derivatives

- Grad, Div, Curl, are all linear operators

$$\vec{\nabla}\left(c_1 f_1(\vec{r}) + c_2 f_2(\vec{r})\right) = c_1 \vec{\nabla} f_1 + c_2 \vec{\nabla} f_2$$

$$\vec{\nabla}\cdot\left(c_1 \vec{A}(\vec{r}) + c_2 \vec{B}(\vec{r})\right) = c_1 \vec{\nabla}\cdot\vec{A} + c_2 \vec{\nabla}\cdot\vec{B}$$

etc.

- Product rules:

Ordinary derivative $\frac{d}{dx}\left(f(\vec{x})g(\vec{x})\right) = f\frac{dg}{dx} + g\frac{df}{dx}$

Must consider many different kinds of products

$$\underbrace{f(\vec{r})\,g(\vec{r})}_{\text{scalar}}, \quad \underbrace{f(\vec{r})\,\vec{A}(\vec{r})}_{\text{vector}}, \quad \underbrace{\vec{A}(\vec{r})\cdot\vec{B}(\vec{r})}_{\text{scalar}}, \quad \underbrace{\vec{A}(\vec{r})\times\vec{B}(\vec{r})}_{\text{Vector}}$$

There are lots of product rules (see G.)

<u>Example:</u> $\vec{\nabla}\cdot\left(f(\vec{r})\,\vec{A}(\vec{r})\right) =$

$$= \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

$$= \left(\frac{\partial f}{\partial x}A_x + \frac{\partial f}{\partial y}A_y + \frac{\partial f}{\partial z}A_z\right)$$

$$+ f\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$\Rightarrow \boxed{\vec{\nabla}\left(f(\vec{r})\vec{A}(\vec{r})\right) = (\vec{\nabla} f)\cdot\vec{A} + f\,\vec{\nabla}\cdot\vec{A}}$$

Second derivatives

Lots of examples:

$\vec{\nabla}\cdot\vec{\nabla}f$, $\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$, $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$, $\vec{\nabla}\times(\vec{\nabla}f)$ etc.

Let's look at first one:

$$\vec{\nabla}\cdot\vec{\nabla}f = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

(Use formula in Cart. coords)

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{\nabla}f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f}$$

Important "second order" differential operator

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}_{\text{in Cartesian coords.}}$$

Note $\nabla^2\vec{A}(\vec{r}) = (\vec{\nabla}\cdot\vec{\nabla})\vec{A}(\vec{r})$

$$= (\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$$

Again, these formulas become <u>very</u> ugly in curvilinear coords

(see G. inside Cover)

Some important general rules

* Theorem: $\vec{\nabla} \times (\vec{\nabla} f(\vec{r})) = 0 \quad \forall f(\vec{r})$ well behaved

Curl of a gradient is always zero

Proof: Consider x-component

$$\left[\vec{\nabla} \times (\vec{\nabla} f(\vec{r})\right]_x = \frac{\partial}{\partial y}(\vec{\nabla} f)_z - \frac{\partial}{\partial z}(\vec{\nabla} f)_y$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 0 \quad \checkmark$$

Same true for y- and z-components

* theorem: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \quad \forall \vec{A}(\vec{r})$

Divergence of curl is zero

Proof left for reader.

We will understand the geometrical meaning of these two theorems in the next lecture.