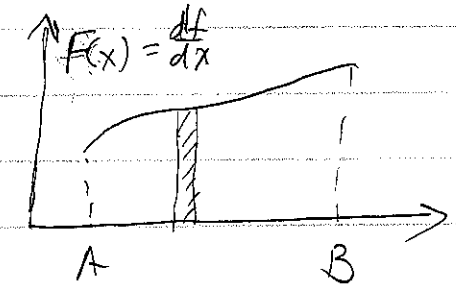
UNM Physics 405: Lecture 3

Integral Vector Calculus

- **Fundamental Theorem of Integral Calculus**

$$\int_a^b F(x)\,dx = f(a) - f(b) \quad \text{where} \quad \frac{df}{dx} = F(x)$$



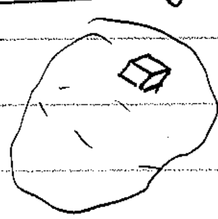
$$\int_a^b F(x)\,dx = \int_a^b \frac{df}{dx}\,dx = \int_{f(a)}^{f(b)} df$$

[The integral of $\frac{df}{dx}$ is determined by the value of f at the endpoints (boundary) of line]

The fundamental theorem generalizes in higher dim.

- **Regions of integration**

- Volume



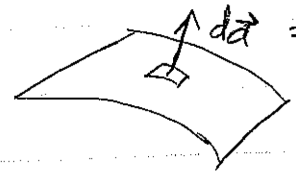
$$\underset{V}{\int} \underset{\substack{\uparrow \\ \text{Scalar} \\ \text{field}}}{F(\vec{r})}\, \underset{\substack{\uparrow \\ \text{Volume} \\ \text{element}}}{d^3r} = \underset{V}{\int} \underset{\text{cartesian}}{F(x,y,z)\, dx\,dy\,dz}$$

In curvilinear coordinates

$$d^3r = ds_r\, ds_\theta\, ds_\phi = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$$

$$= r^2 \sin\theta\, dr\, d\theta\, d\phi \quad = -r\,dr\,d(\cos\theta)\,d\phi$$

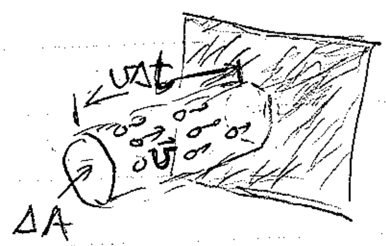
• Surface (with "orientation")



$$d\vec{a} = \hat{n}\, da = \begin{cases} \text{magnitude} = \text{area of patch} \\ \text{direction} = \text{normal to surface in choose direction} \end{cases}$$

Flux and surface integrals

Consider a uniformly flowing fluid with mass density ρ, moving with velocity $\vec{v}$. Consider a hole of area ΔA cut in a surface oriented $\perp$ to $\vec{v}$

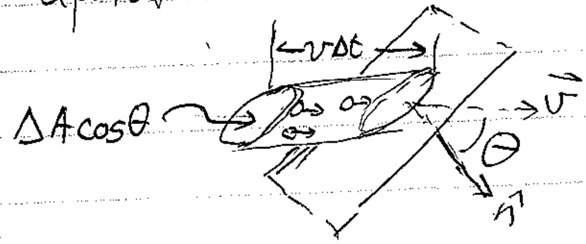


Mass that flows through hole in time Δt

$$\Delta M = \underbrace{\rho\, v}_{=\text{ flux}}\, \Delta t\, \Delta A$$

Can define $\vec{J} = \frac{\Delta M}{\Delta t\, \Delta A} = \rho \vec{v} =$ Rate of flow of mass / A

If aperature is not oriented normal to $\vec{v}$



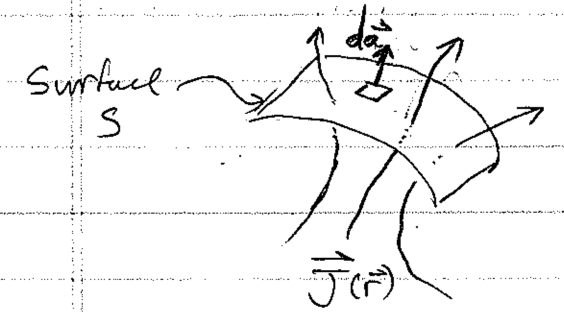
$$\Rightarrow \Delta M = \rho v \Delta t \Delta A \cos\theta$$

$$\Rightarrow \frac{\Delta M}{\Delta t} = \rho \vec{v} \cdot (\vec{\Delta A})$$

$$\vec{\Delta A} = \hat{n}\, \Delta A$$

$\Rightarrow$ Flux is the dot product of $\vec{J}$ with $\vec{\Delta A}$

For an arbitrary surface S

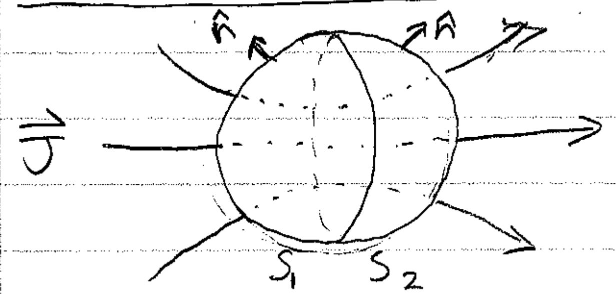


Flux of $\vec{J}$ through patch $d\vec{a}$ at $\vec{r}$ $= \vec{J}(\vec{r}) \cdot d\vec{a}(\vec{r})$

$\Rightarrow$ Flux of $\vec{J}$ through S

$$\text{Flux} = \int_S \vec{J}(\vec{r}) \cdot d\vec{a}(\vec{r})$$

Closed Surfaces

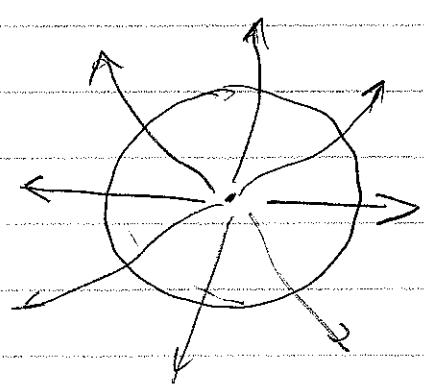


$$\text{Flux} = \oint_S \vec{J} \cdot d\vec{a}$$

$$= \underbrace{\int_{S_1} \vec{J} \cdot d\vec{a}}_{\text{Flux in}} + \underbrace{\int_{S_2} \vec{J} \cdot d\vec{a}}_{\text{Flux out}}$$

Here, no sources inside S (and incompressible fluid)

$$\Rightarrow \oint \vec{J} \cdot d\vec{A} = 0$$



$$\Rightarrow \text{Flux } \oint_S \vec{J} \cdot d\vec{a} \neq 0$$

Net flow out

Here source inside S

Example: $\vec{J}(\vec{r}) = J_0 \vec{r}$. What is the flux of $\vec{J}$ through a sphere of radius R_0 centered at the origin.

$$\text{Flux} = \oint \vec{J} \cdot d\vec{a} = \oint \vec{J}(\vec{r} = R_0 \hat{r}) \cdot \hat{r}\, da$$

$$= \oint (J_0 R_0)\, da = J_0 R_0 \oint da = (J_0 R_0)\, 4\pi R_0^2$$

$$= J_0\, 4\pi R_0^3$$

This is non-zero as the fluid flows from origin and surface enclosed origin

We would like to define a quantity which tells us if there is a "source" at a point. Note, that in the example above, the Flux depended on the radius of the surface, R_0. If we divide by volume enclosed, we get a result that depends only on the field $\vec{J}$

$\Rightarrow$ Define

$$\boxed{\operatorname{div} \vec{J}(\vec{r})\Big|_{\vec{r}_0} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{J} \cdot d\vec{a}}{\Delta V}}$$

where S is a surface centered @ $\vec{r}$ and ΔV is the enclosed volume

For the simple example, $\vec{J} = J_0 \vec{r}$, $\Delta V = \frac{4\pi}{3} R_0^3$

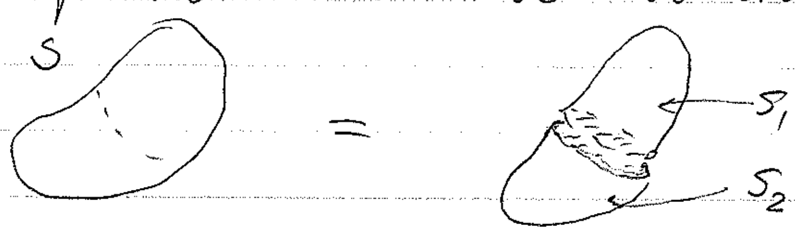
$$\Rightarrow \lim_{\Delta V \to 0} \frac{\oint \vec{J} \cdot d\vec{a}}{\Delta V} = 3 = \vec{\nabla} \cdot \vec{J}$$

This is a general result: $\boxed{\operatorname{div} \vec{J} = \lim \frac{\oint \vec{J} \cdot d\vec{a}}{\Delta V} = \vec{\nabla} \cdot \vec{J}}$

Geometrically: $\vec{\nabla} \cdot \vec{J}$ = local "divergence" of $\vec{J}$

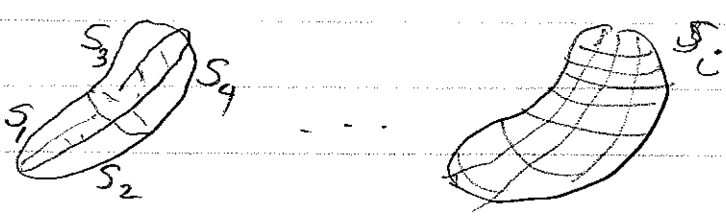
<u>Fundamental theorem for Surface Integrals</u>

Take a volume V bounded by surface S. We can partion into two volumes V_1 and V_2



$$\Rightarrow \oint_S \vec{F} \cdot d\vec{a} = \oint_{S_1} \vec{F} \cdot d\vec{a} + \oint_{S_2} \vec{F} \cdot d\vec{a}$$, since flux through the parition out of S_1 is cancelled by flux through pardition in S_2

Further subdivide



Flux cancels every where except on boundary

Goes to div $\vec{F}$ in limit

$$\Rightarrow \oint_S \vec{F}\cdot d\vec{a} = \sum_{i=1}^{N} \oint_{S_i} \vec{F}\cdot d\vec{a} = \sum_{i=1}^{N} \Delta V_i \boxed{\frac{\oint \vec{F}\cdot d\vec{a}_i}{\Delta V_i}}$$

Now take the limit as $N \to \infty$

$$\boxed{\oint_S \vec{F}\cdot d\vec{a} = \int_V (\vec{\nabla}\cdot\vec{F})\, d^3r}$$

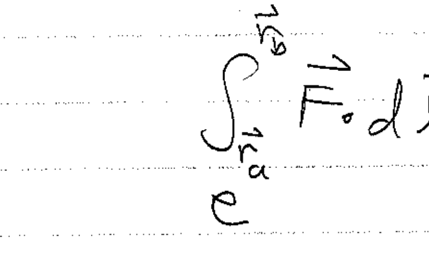
$\Rightarrow$ Total flux through a closed surface = Total # of sources inside the enclosed volume

Note general form

$\vec{F}$ on boundary $= \int_{\text{region enclosed by boundary}}$ (derivative of $\vec{F}$)

Line Integrals and Circulation

Curve embedded in a higher dim space



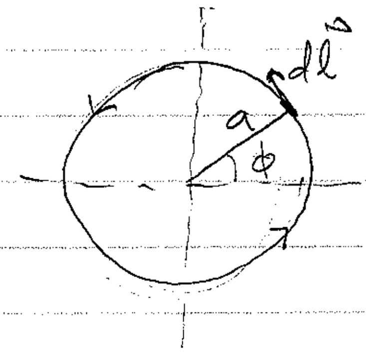
Curve w/ a direction

$$d\vec{l} = \begin{cases} \text{mag} = \text{length element} \\ \text{direction} = \text{tangent to } C \end{cases}$$

$$\int_{\vec{r}_a\ (C)}^{\vec{r}_b} \vec{F}\cdot d\vec{l} =$$ dot of local $\vec{F}$ with local $d\vec{l}$ added up along C

Example $\vec{F} = F_0(-y\hat{x} + x\hat{y}) = F_0 \rho \hat{\phi}$

let C = closed circle radius a, counter clock wise, in x-y plane



$d\vec{l} = a\,d\phi\,\hat{\phi} \qquad \vec{F}\Big|_C = F_0 a \hat{\phi}$

$\Rightarrow \oint_C \vec{F}\cdot d\vec{l} = \int_0^{2\pi} d\phi\, F_0 a^2$

$= 2\pi a^2 F_0$

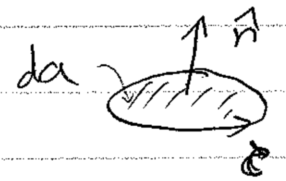
For closed curve: $\oint \vec{F}\cdot d\vec{l} \equiv$ circulation of $\vec{F}$

Curl:

We would like a <u>local</u> measure of circulation of $\vec{F}$ at a point

Define: $\hat{n}\cdot(\text{curl}\,\vec{F}) = \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_C \vec{F}\cdot d\vec{l}$

Where ΔA is the area bounded by curve C and $\hat{n}$ is the normal to ΔA, by right-hand-rule



For our example: $\vec{F} = F_0 \rho \hat{\phi}$

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_C \vec{F} \cdot d\vec{l} = \lim_{a \to 0} \left(\frac{1}{\pi a^2}\right)\left(2\pi a^2 F_0\right)$$

$$= 2F_0$$

Compare to $\vec{\nabla} \times \vec{F} = F_0 \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix}$

$$\Rightarrow \vec{\nabla} \times \vec{F} = F_0 \hat{z}\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) = 2F_0 \hat{z}$$

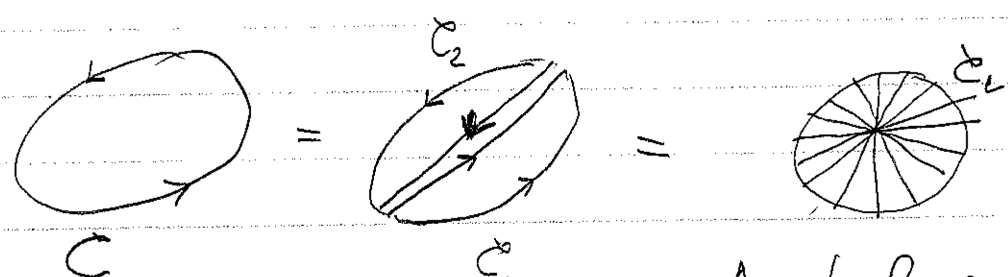
$$\Rightarrow \hat{z} \cdot (\vec{\nabla} \times \vec{F}) = \lim_{\Delta A \to 0} \frac{\oint_C \vec{F} \cdot d\vec{l}}{\Delta A}$$

$$\boxed{\text{general resul } \hat{n} \cdot \vec{\nabla} \times \vec{F}\Big|_{\vec{r}_0} = \lim_{\Delta A \to 0} \frac{\oint \vec{F} \cdot d\vec{l}}{\Delta A}}$$

<u>Stokes theorem</u>: $\boxed{\oint \vec{F} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{a}}$

circulation around closed loop = Total "swirl" of field inside region = flux of curl

Proof



As before