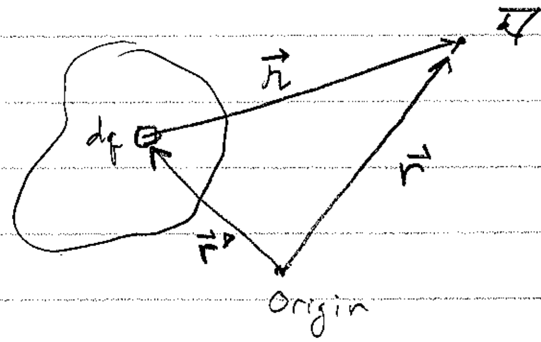
UNM Physics 405: Lecture 6: Calculation of $\vec{E}$-fields.

General form of electric field generated by a charge distribution $\rho(\vec{r})$

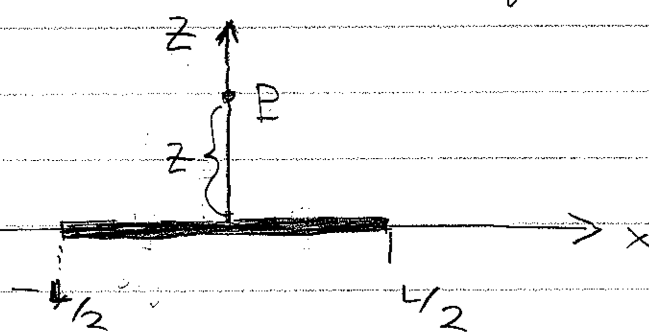


$dq = \rho(\vec{r}')\, d^3r'$

$$\Rightarrow \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{\mathscr{r}^3}\vec{\mathscr{r}}$$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\, \rho(\vec{r}')\, \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}}$$

Example: Electric field a distance z above the midpoint of a line segment, length L, carrying uniform line charge λ = charge/length

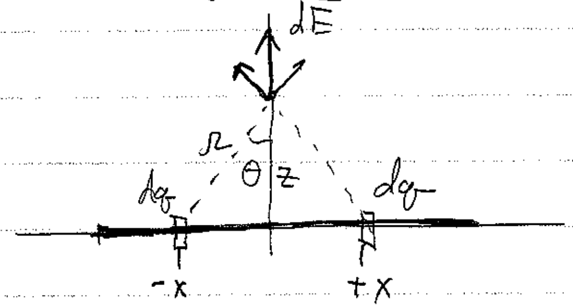


$$\vec{E}(P) = \int_{-L/2}^{L/2} dx \frac{\lambda\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}$$

The key to solving this problem is to use <u>symmetry</u>

At the midpoint of the line segment the electric field points only along the z direction ($\perp$ to the line segment)

$\Rightarrow$ We need only add the z-components of $\vec{E}$ from symmetrically arranged "chunks" of charge



$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2dq}{r^2}\underbrace{\cos\theta}_{\text{projection onto } z\text{-axis}}\,\hat{z}$$

$$r = \sqrt{x^2+z^2}\ ,\quad \cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+z^2}}$$

$$\Rightarrow \vec{E}(z) = \int d\vec{E} = \left[\frac{1}{4\pi\epsilon_0}\int_0^{L/2} dx\,\frac{2\lambda}{(x^2+z^2)}\,\frac{z}{\sqrt{x^2+z^2}}\right]\hat{z}$$

$$= \hat{z}\,\frac{1}{4\pi\epsilon_0}\,2\lambda z\int_0^{L/2} dx\,\frac{1}{(x^2+z^2)^{3/2}}$$

$$= \hat{z}\,\frac{1}{4\pi\epsilon_0}\,\frac{2\lambda}{z}\underbrace{\int_0^{L/2z} d\left(\frac{x}{z}\right)\frac{1}{\left(1+\left(\frac{x}{z}\right)^2\right)^{3/2}}}$$

$$\Rightarrow \quad \vec{E}(z) = \hat{z}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{2\lambda}{z}\underbrace{\int_0^{\frac{L}{2z}}\frac{du}{(1+u^2)^{3/2}}}_{I}$$

Aside:

Let $u = \tan\theta = \frac{x}{z} \Rightarrow du = \sec^2\theta\, d\theta$

$$1+u^2 = 1+\tan^2\theta = \sec^2\theta$$

$$\Rightarrow \int_0^{L/2z}\frac{du}{(1+u^2)^{3/2}} = \int_0^{\tan^{-1}(L/2z)}\frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int_0^{\tan^{-1}(L/2z)}\cos\theta\, d\theta$$

$$= \sin\theta\Big|_0^{\tan^{-1}\left(\frac{L}{2z}\right)} = \frac{L/2}{\left(z^2+\frac{L^2}{2}\right)^{1/2}}$$

$$\Rightarrow \quad \boxed{\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\frac{L/2}{\left(z^2+\frac{L^2}{4}\right)^{1/2}}\hat{z}}$$

* Check Limit:

- As $z\to\infty$, i.e. $z \gg L$, i.e. $z/L \to \infty$

$$\vec{E}(z) \to \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\frac{L/2}{z}\hat{z}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{(\lambda L)}{z^2}\hat{z} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{(Q_{total})}{z^2}\hat{z}}$$

Field of a point charge!

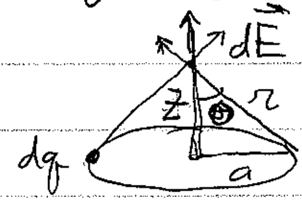
- As $z\to 0$ i.e. $L \gg z$, i.e. $L/z \to \infty$

$$\boxed{\vec{E}(z) \to \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\hat{z}}$$

Field of infinite line charge

Example: Ring Charge

Ring of charge Q, radius a,
Find $\vec{E}$ distance z above center



$\cos\theta = \frac{d}{r}$ $\qquad r = \sqrt{z^2+a^2}$

Symmetry: Only "z" component of $\vec{E}$ survives

$$dE_z = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \cos\theta = \frac{1}{4\pi\epsilon_0} (dq) \frac{z}{r^3}$$

$$\text{Total } \vec{E}(P) = \hat{z}\int dE_z = \hat{z} \frac{1}{4\pi\epsilon_0} \frac{z}{(z^2+a^2)^{3/2}} \int_{\text{ring}} dq$$

$$= \hat{z} \frac{Q}{4\pi\epsilon_0} \frac{z}{(z^2+a^2)^{3/2}}$$

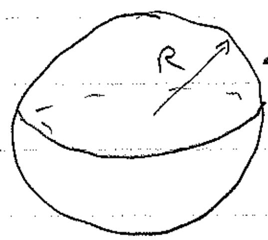
Limit $z \gg a \Rightarrow$ Point charge

$$\vec{E}(P) = \hat{z} \frac{Q}{4\pi\epsilon_0} \frac{z}{(z^2)^{3/2}} = \hat{z} \frac{1}{4\pi\epsilon_0} \frac{Q}{z^2} \checkmark$$

$z \ll a \Rightarrow$ Field goes to zero

$$\vec{E}(P) = \hat{z} \frac{Q}{4\pi\epsilon_0} \frac{z}{a^2} \rightarrow 0 \text{ as } z \rightarrow 0$$

Field of a uniformly charged sphere, charge Q



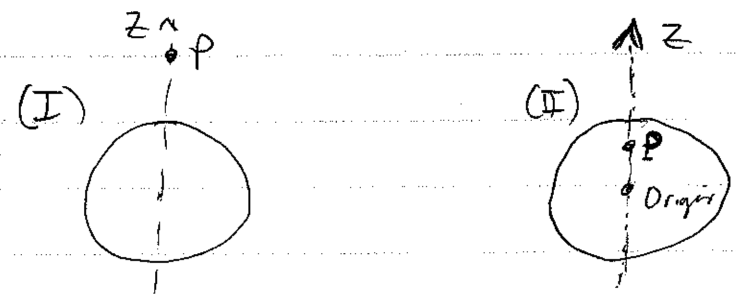
surface density

$$\sigma = \frac{Q}{4\pi R^2}$$

<u>Symmetry</u>: • $|\vec{E}(\vec{r})|$ depends only on $|\vec{r}|$ (spherically symmetric)

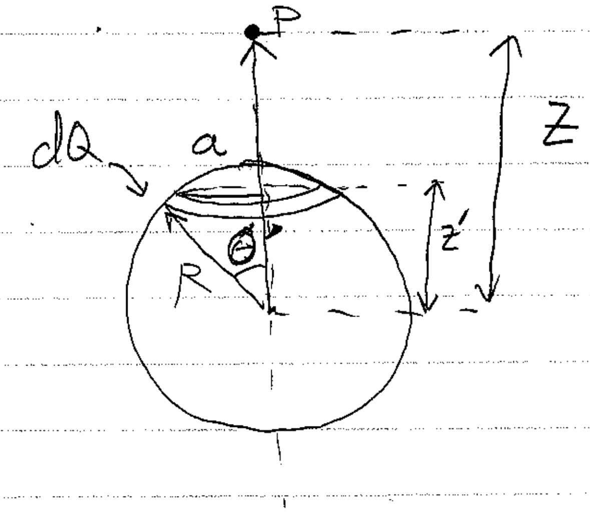
• Direction of $\vec{E}(\vec{r})$ is $\hat{r}$

$\Rightarrow$ Can determine $\vec{E}$-field on an axis and rotate



Two cases: Inside and Outside Sphere

To solve: Break up sphere into rings



$$a = R\sin\theta$$

$$dQ = d\lambda \, 2\pi a$$

$$d\lambda = \sigma(R\,d\theta)$$

$$z' = R\cos\theta$$

$$d\vec{E}(P) = \hat{z}\,\frac{dQ}{4\pi\epsilon_0}\left[\frac{z-z'}{\left((z-z')^2+a^2\right)^{3/2}}\right]$$

$$= \hat{z}\,\frac{2\pi\sigma R^2}{4\pi\epsilon_0}\,\frac{(z-R\cos\theta')\sin\theta'\,d\theta'}{\left[(z-R\cos\theta')^2+(R\sin\theta')^2\right]^{3/2}}$$

$$= \hat{z}\,\frac{2\pi\sigma R^2}{4\pi\epsilon_0}\,\frac{(z-R\cos\theta')\,d\theta'}{\left[z^2+R^2-2Rz\cos\theta'\right]^{3/2}}$$

$$\Rightarrow \vec{E}(P) = \hat{z}\,\frac{2\pi\sigma R^2}{4\pi\epsilon_0}\int_0^{\pi} d\theta'\,\frac{(z-R\cos\theta')\sin\theta'}{\left[z^2+R^2-2Rz\cos\theta'\right]^{3/2}}$$

Yikes!

Let $\mu = \cos\theta' \Rightarrow d\mu = -\sin\theta' d\theta'$

$$\Rightarrow \vec{E}(\vec{P}) = \hat{z}\,\frac{2\pi\sigma R^2}{4\pi\epsilon_0}\int_{-1}^{1} d\mu\,\frac{(z-\mu R)}{(z^2+R^2-2\mu R)^{3/2}}$$

$$= \hat{z}\,\frac{2\pi\sigma R^2}{4\pi\epsilon_0}\left[\frac{\mu z - R}{z^2\sqrt{z^2+R^2-2\mu R}}\right]_{-1}^{1}$$

$$= \hat{z}\,\frac{2\pi\sigma R^2}{4\pi\epsilon_0 z^2}\left[\frac{z-R}{\sqrt{(z-R)^2}} - \frac{-z-R}{\sqrt{(z+R)^2}}\right]$$

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Now we must consider two case

(I) $z > R$ (outside sphere)

$$\Rightarrow \sqrt{(z-R)^2} = z-R$$

$$\Rightarrow \vec{E}(\vec{P}) = \hat{z}\,\frac{2\pi\sigma R^2}{4\pi\epsilon_0 z^2}\Big(1-(-1)\Big) = \hat{z}\,\frac{4\pi R^2\sigma}{4\pi\epsilon_0 z^2}$$

$$= \hat{z}\,\frac{1}{4\pi\epsilon_0}\,\frac{Q}{z^2}$$

(II) $z < R$ (inside sphe

$$\Rightarrow \sqrt{(z-R)^2} = \sqrt{(R-z)^2} = R-z$$

$$\Rightarrow \vec{E}(\vec{P}) = \hat{z}\,\frac{2\pi\sigma R^2}{4\pi\epsilon_0 z}\big(-1-(-1)\big) = 0$$

Thus, using spherical symmetry we have

$$\vec{E}(\vec{r}) = \begin{cases} 0 & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} & r > R \end{cases}$$

Note: For a sphere, E-field exactly that of a point charge at origin outside sphere