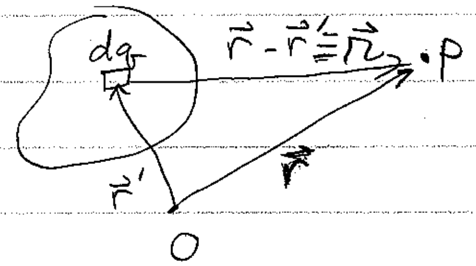
UNM Physics 405: Lecture 7: Gauss' Law

In principle, we are done with electrostatics



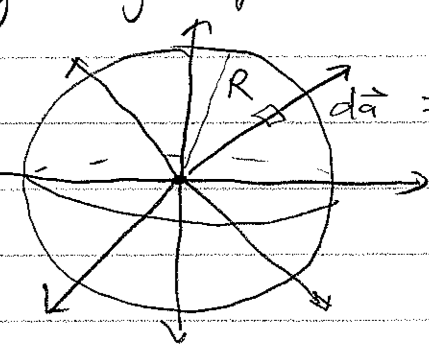
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\rho(\vec{r}')\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

These integrals are generally very nasty (e.g. sphere of charge). We need a bag of tricks.

Integral form of Gauss' Law

Consider the flux of the electric field associated with a point charge through a "imaginary sphere" with center @ q.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$



$$d\vec{a} = \hat{r}\, r^2 d\Omega$$

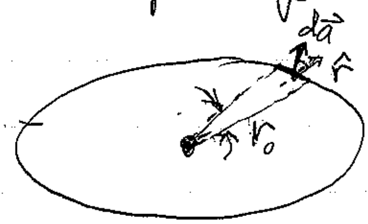
↖ element of solid angle

$$d\Omega = \sin\theta\, d\theta\, d\phi = -d(\cos\theta)\, d\phi$$

$$\oint_{\text{sphere}} \vec{E}\cdot d\vec{a} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\underbrace{\oint da}_{4\pi R^2} = \frac{q}{\epsilon_0}$$

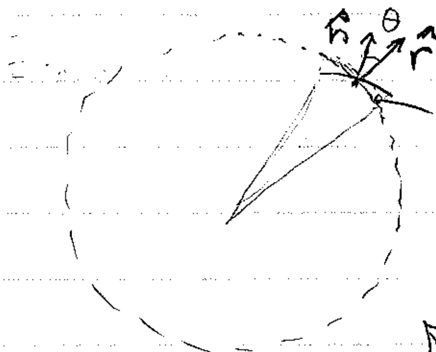
Independent of R!

In fact, the flux of $\vec{E}$ is independent of the shape of S.



$$d\vec{a} = \hat{n}\, da$$

$$\vec{E}\cdot d\vec{a} = \frac{1}{4\pi\epsilon_0}\frac{q}{r_0^2}\underbrace{(\hat{r}\cdot\hat{n})}_{\cos\theta} da$$



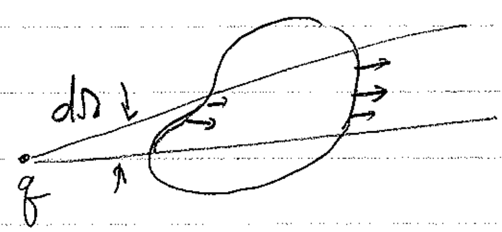
$$da' = da\cos\theta = r_0^2\, d\Omega \cos\theta$$

$$\Rightarrow da = \frac{r_0^2\, d\Omega}{\cos\theta}$$

$$\Rightarrow \boxed{\vec{E}\cdot d\vec{a} = \frac{d\Omega}{4\pi}\frac{q}{\epsilon_0}}$$

$\Rightarrow \oint \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0}$ if q is inside surface

If q is outside surface



Since flux independent of shape,

flux in = flux out

$$\Rightarrow \oint \vec{E}\cdot d\vec{a} = 0$$

Superposition Principle

Given N-charges inside a closed surface

$$\oint_S \vec{E}\cdot d\vec{a} = \int_S (\vec{E}_1 + \vec{E}_2 + \cdots \vec{E}_N)\cdot d\vec{a} = \sum_{i=1}^{N} \frac{q_i}{\epsilon_0} = \frac{Q_{enclosed}}{\epsilon_0}$$

Gauss's Law: The Flux of $\vec{E}$ through any closed surface $= \left(\frac{1}{\epsilon_0}\right) \times$ (the total charge enclosed inside)

$$\boxed{\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enclosed}}{\epsilon_0}}$$

- Gauss's Law is a consequence of the $\frac{1}{r^2}$ nature of the force $\Rightarrow$ geometrical.
  The "number of field lines piercing out of surface" = # of sources inside the volume.

- Gauss's Law is always true, but it not always useful as a tool for calculating the $\vec{E}$-field.

> Use Gauss's law for charge distributions which have a high degree of symmetry

Examples

(1) Sphere of charge of radius R which has a charge Q uniformly distributed throughout

Symmetry! Spherical $\Rightarrow$ $\vec{E}(\vec{r}) = E(r)\hat{r}$

- $\vec{E}$ points in radial direction
- $|\vec{E}|$ depends only on r (not θ and ϕ)

Choose a "Gaussian surface" so that $\vec{E}\cdot\hat{n}\,da$ is constant of S. Here that is a sphere

- $r > R$

$$\oint \vec{E}\cdot d\vec{a} = E(r)\oint da$$

$$= E(r)\,4\pi r^2 = \frac{Q_{enc}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}}$$ (field of point charge Q @ origin)

$r > R$

- $r < R$

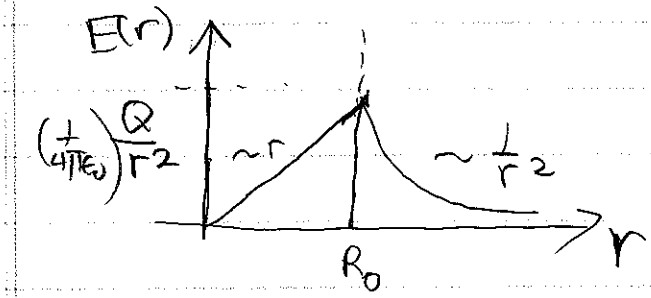
$$E(r)\,4\pi r^2 = \frac{Q_{enc}(r)}{\epsilon_0}$$

$$Q_{enc}(r) = \left(\frac{4}{3}\pi r^3\right)\rho = \frac{4}{3}\pi r^3\,\frac{Q}{\frac{4}{3}\pi R^3} = Q\,\frac{r^3}{R^3}$$

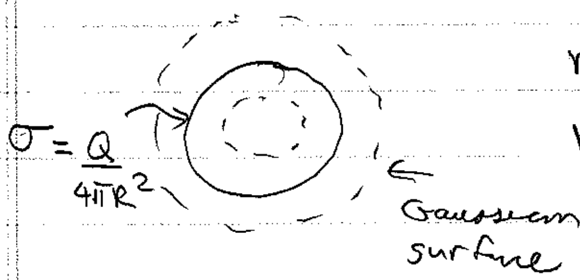
$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\,Q\,\frac{r}{R^3}\,\hat{r}}$$

$r < R$

(1) Sphere of charge continued

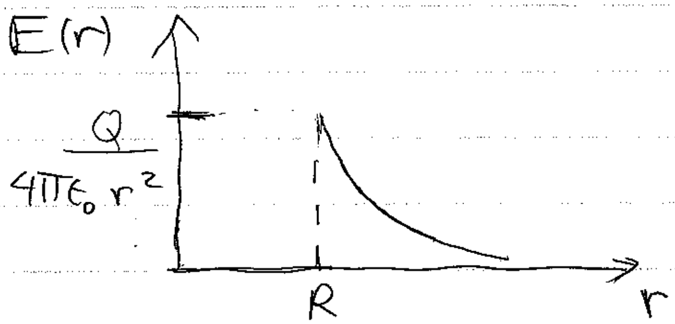


(2) Spherical shell of charge



$r<R \quad Q_{enc} = 0 \Rightarrow E = 0$

$r>R \quad Q_{enc} = Q \Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}$



The field inside the sphere is zero.

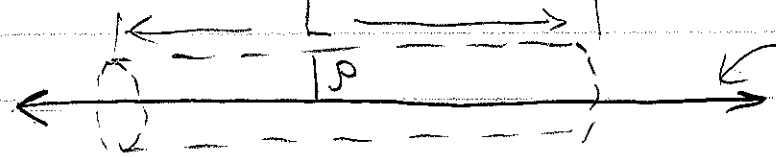
* Note * This would not be true without symmetry. We used the fact that

$$\oint \vec{E}\cdot d\vec{a} = E\big|_{surface} \oint d\vec{a}$$

We can have $\oint \vec{E}\cdot d\vec{a} = 0$ w/o

$\vec{E} = 0$ in non-symmetric case

Example 3: Infinite line charge



$\lambda = \frac{\text{charge}}{\text{length}}$

Here we know $\vec{E} = E(\rho)\hat{\rho}$ (Cylindrical symmetry)

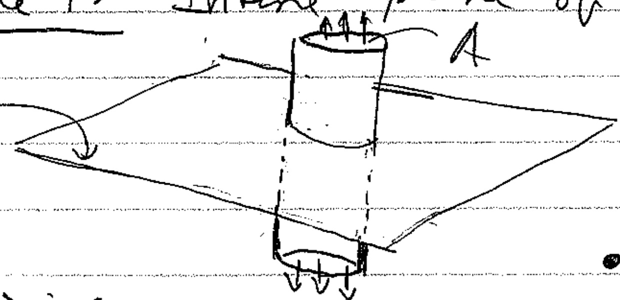
Choose Gaussian surface as cylinder, shown above

$$\oint_S \vec{E}\cdot da = \underset{\substack{\text{surface}\\ \text{excluding "caps"}}}{\int} E(\rho)\, da = E(\rho)\, 2\pi\rho L$$

$$= \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

$$\Rightarrow \vec{E}(\rho) = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\rho}}{\rho}$$ falls off as $\frac{1}{\rho} = \frac{1}{\sqrt{x^2+y^2}}$

Example 4: Infinite plane of charge



$\sigma = \frac{\text{charge}}{\text{area}}$

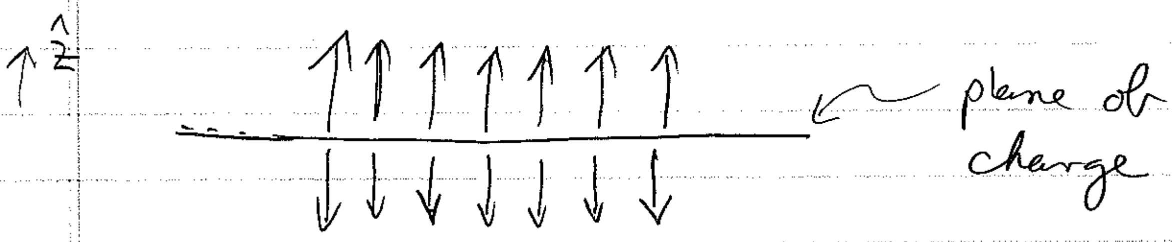
Symmetry
- Direction $\perp$ to plane
- Magnitude independent of distance from plane

$$\Rightarrow \vec{E} = \{E_0\hat{z} \text{ above}, \underset{\text{below}}{-E_0\hat{z}}\}$$

$$\Rightarrow \oint \vec{E}\cdot d\vec{A} = E_0(A_{top} + A_{bottom}) = 2AE_0$$

$$= \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow \quad \vec{E} = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\hat{z} & \text{above} \\ -\dfrac{\sigma}{2\epsilon_0}\hat{z} & \text{below} \end{cases}$$



Gauss's Law: Differential form

Use the divergence theorem

$$\oint_S \vec{E}\cdot d\vec{a} = \int_V (\vec{\nabla}\cdot\vec{E})\, d^3r = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V \rho(\vec{r})\, d^3r$$

Volume arbitrary $\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0}}$

Meaning: The local divergence of $\vec{E}$ is given by the local source of electric field @ $\vec{r}$, $\rho(\vec{r})$

The integral theorem $\oint \vec{E}\cdot d\vec{a} = \dfrac{Q_{enc}}{\epsilon_0}$ is a global statement