UNM Physics 405: Lecture 8

Vector equations of electrostatics

Our expression for Gauss's Law: $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

is global — an expression for the electric field over a surface. We can turn this into a local expression using the fundamental theorems of integral calculus.

Using the divergence theorem:

$$\oint_S \vec{E}\cdot d\vec{a} = \int_{V_{enc}} (\vec{\nabla}\cdot\vec{E})\, d^3r$$

Where V_{enc} is the volume enclosed by S

Now $Q_{enc} = \int_{V_{enc}} \rho(\vec{r})\, d^3r$ is the charge enclosed inside S

Since V_{enc} is arbitrary,

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}}$$

Differential form of Gauss's Law

This is a local statement at position $\vec{r}$. It says that the local $\vec{\nabla}\cdot\vec{E}$ @ $\vec{r}$ is equal to the local charge density (source) @ $\vec{r}$.

Check in a few cases

Ex: • Uniform ball of charge of radius R

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0} \dfrac{\vec{r}}{R^3} & r \leq R \\ \dfrac{Q}{4\pi\epsilon_0} \dfrac{\hat{r}}{r^2} & r \geq R \end{cases}$$

Inside $r \leq R$: $\vec{\nabla} \cdot \vec{E} = \dfrac{Q}{4\pi\epsilon_0 R^3} \vec{\nabla} \cdot \vec{r}$, $\quad \vec{\nabla} \cdot \vec{r} = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 3$

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{3Q}{4\pi R^3 \epsilon_0} = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right) \frac{1}{\epsilon_0} = \frac{\rho_{inside}}{\epsilon_0} \checkmark$$

Outside $r \geq R$: $\vec{\nabla} \cdot \vec{E} = \dfrac{1}{r^2} \dfrac{\partial}{\partial r}(r^2 E_r)$ (using div in sph. coords)

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_0}\right) = 0 \checkmark$$

since $\rho = 0$ outside the ball

Ex: • Point charge @ origin

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \qquad r \geq R$$

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r^2}(r^2 E_r) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(\frac{q}{4\pi\epsilon_0}\right) = 0$$

This is true everywhere except @ origin.

The problem is that $\vec{\nabla} \cdot \vec{E}$ is singular (blows up) @ origin since charge density blows up there.

Curl of $\vec{E}$ and definition of electrostatic potential

Any vector field is determined by both its divergence and curl.

We found $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$. What about $\vec{\nabla}\times\vec{E}$?

It suffices to find $\vec{\nabla}\times\vec{E}$ for a point charge (by superposition)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = E_r(r)\,\hat{r}$$

Curl in spherical coords: (when $\vec{v}(\vec{r}) = v_r(\vec{r})\,\hat{r}$)

$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}\hat{\theta} - \frac{1}{r}\frac{\partial}{\partial\theta}v_r\,\hat{\phi}$$

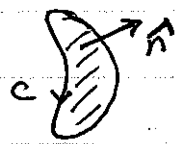
$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0 \text{ for a point charge (static)}}$$

Note, unlike $\vec{\nabla}\cdot\vec{E}$, $\vec{\nabla}\times\vec{E} = 0$ is truly zero everywhere — no singularity at origin

By principle of superposition, given any static charge distribution $\rho(\vec{r})$:

$$\boxed{\vec{\nabla}\times\vec{E} = 0} \text{ for electrostatics}$$

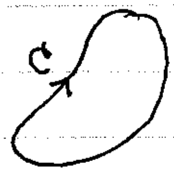
Implication for Stokes's Law



$$\oint_C \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{l} = 0$$ ← electrostatics

$$\Rightarrow \boxed{\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} \text{ is independent of path connecting } \vec{a} \text{ and } \vec{b}}$$

Why?



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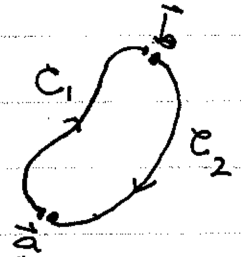


Closed curve = Sum of two directed curves with same end points

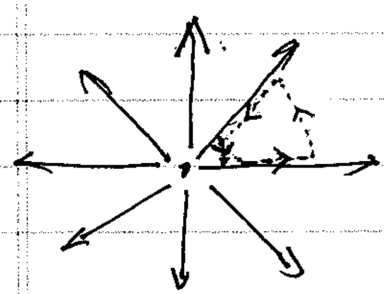
$$\Rightarrow \oint_C \vec{E}\cdot d\vec{l} = \int_{\vec{a}\,C_1}^{\vec{b}} \vec{E}\cdot d\vec{l} + \int_{\vec{b}\,C_2}^{\vec{a}} \vec{E}\cdot d\vec{l}$$

$$= \int_{\vec{a}\,C_1}^{\vec{b}} \vec{E}\cdot d\vec{l} - \int_{\vec{a}\,C_2}^{\vec{b}} \vec{E}\cdot d\vec{l} = 0$$

in electrostatics $\Rightarrow$

$$\boxed{\int_{\vec{a}\,C_1}^{\vec{b}} \vec{E}\cdot d\vec{l} = \int_{\vec{a}\,C_2}^{\vec{b}} \vec{E}\cdot d\vec{l} \text{ for } \underline{\text{any}} \text{ two curves connecting } \vec{a} + \vec{b}}$$

Geometrical picture



θ and ϕ components are zero, radial component only varies with $r \Rightarrow$ Central field

Defining the Electrostatic Potential

Because $\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$ is independent of path, we can define a single valued scalar field

$$\boxed{V(\vec{r}) \equiv \underset{\substack{\uparrow \\ \text{(convention)}}}{-} \int_{\underset{\text{(any path)}}{\vec{r}_0}}^{\vec{r}} \vec{E}(\vec{r}\,') \cdot d\vec{l} \qquad \text{where } V(\vec{r}_0) = 0, \quad \vec{r}_0 \equiv \text{ground}}$$

The choice of ground, $\vec{r}_0$, is arbitrary.

Usually, for a charge distribution confined to a finite region of space, one take $\vec{r}_0$ at ∞.

Careful: for infinitely distributed charge distributions e.g. plane or line, one cannot do this. Such distributions, of course, are not physical, but approximations to real distributions.

Units: $[V] \equiv \text{Volt} \qquad [E] \equiv \dfrac{\text{Volt}}{\text{meter}}$