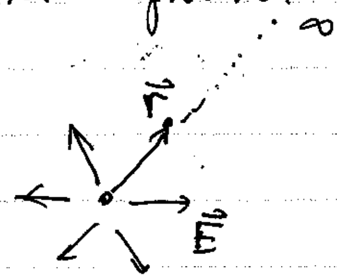
UNM Physics 405: Lecture 9

Examples of Electrostatic Potential

- Potential of a point charge @ origin with ground @ ∞



Take path along radial direction

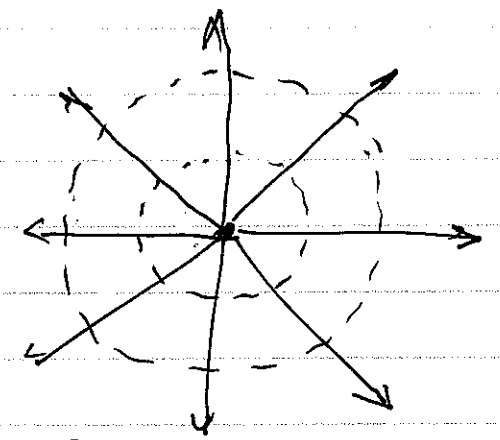
$$d\vec{l} = dr\,\hat{r}$$

$$V(r) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l} = -\int_{\infty}^{r} E(r')\,dr'$$

$$\Rightarrow V(r) = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' = \frac{q}{4\pi\epsilon_0}\frac{1}{r'}\Big|_{\infty}^{r}$$

$$\Rightarrow \boxed{V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}}$$

Spherical equipotential surfaces



$\vec{E} \perp$ to $V(r)$

Clue!

Fundamental Integral theorem

$$V(\vec{r}) - \underbrace{V(\vec{r}_0)}_{0} = \int_{\vec{r}_0}^{\vec{r}} (\vec{\nabla}V)\cdot d\vec{l} = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})}$$

$\Rightarrow$ The electrostatic field is the gradient of a scalar

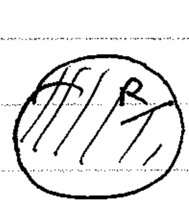
Example of Point charge @ origin

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}|} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$\Rightarrow \vec{E}(\vec{r}) = -\vec{\nabla}V = -\hat{r}\frac{\partial V}{\partial r}$$ (in spherical coords when V independent of θ, ϕ)

$$\Rightarrow \vec{E}(r) = -\hat{r}\frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \hat{r}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \checkmark$$

Example: Sphere of charge with Q distributed throughout volume of ball



$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\begin{cases}\frac{r}{R^3}\hat{r} & r \le R \\ \frac{1}{r^2}\hat{r} & r \ge R\end{cases}$$

Again take radial path.

Consider two cases:

$r > R$ $$V(r) = -\frac{Q}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{dr'}{r'^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

$r < R$ $$V(r) = -\int_{\infty}^{R} E_>(r')\,dr' - \int_{R}^{r} E_<(r')\,dr'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{R} - \frac{Q}{4\pi\epsilon_0}\frac{1}{R^3}\underbrace{\int_{R}^{r} r'\,dr'}_{\frac{1}{2}(r^2 - R^2)}$$

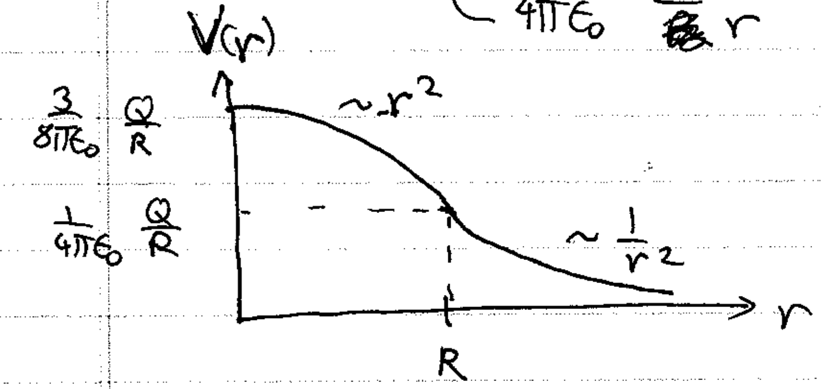
(Next Page)

Thus,

$r<R \qquad V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}\left(1 - \frac{r^2-R^2}{2R^2}\right)$

$\qquad\qquad = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)$

$$\Rightarrow V(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{Q}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right) & r \le R \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r} & r \ge R \end{cases}$$



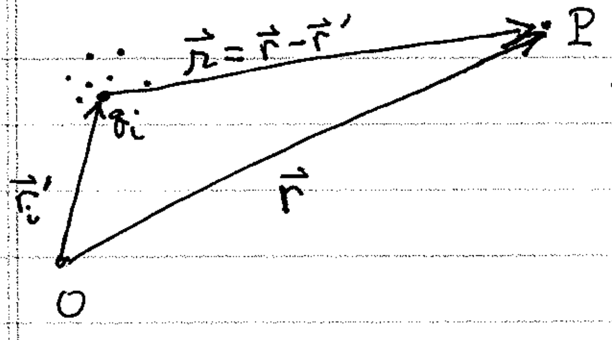
Note: $V(r)$ is everywhere differentiable if $\vec{E}$ is everywhere defined (not singular)

$$-\vec{\nabla} V(r) = -\hat{r}\frac{\partial V(r)}{\partial r} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{r} & r \le R \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} & r \ge R \end{cases}$$

✓ as before

Generally, given an arbitrary charge distribution, it is much easier to solve electrostatic problem by finding $V(\vec{r})$ first, and the $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

For a finite collection of point charges:



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i'|}$$

with ground $\vec{r}_0 = \infty$

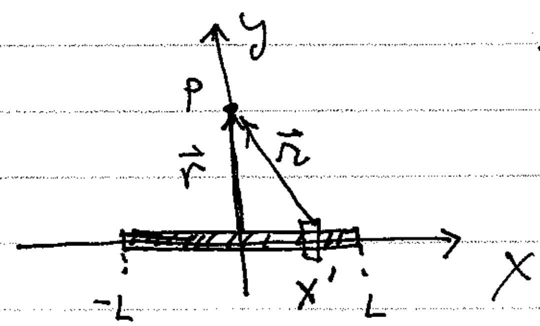
For continuous distribution (in finite region)

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} = \frac{1}{4\pi\epsilon_0} \int_S da' \frac{\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|}, \text{ etc}$$

Remember: The choice of $\vec{r}_0$ is arbitrary. Only potential differences are physical.

Example: Line charge of length $2L$, λ = charge/length

Find potential on the bisector



$$V(y) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\mathscr{r}}$$

$$dq = \lambda dx'$$

$$\mathscr{r} = \sqrt{x'^2 + y'^2}$$

(easy set up)

Thus, $V(y) = \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{dx'}{(x'^2+y'^2)^{1/2}} = \frac{\lambda}{4\pi\epsilon_0}\ln\left[x' + \sqrt{x'^2+y'^2}\right]\Bigg|_{-L}^{L}$

$$\therefore \boxed{V(y) = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L+\sqrt{y^2+L^2}}{-L+\sqrt{y^2+L^2}}\right]}$$

On y-axis: $\vec{E} \doteq -\vec{\nabla}V = -\hat{y}\,\frac{\partial V(y)}{\partial y}$

$$\Rightarrow \vec{E} = \frac{\lambda}{4\pi\epsilon_0}\frac{y}{\sqrt{y^2+L^2}}\left(\frac{1}{L+\sqrt{y^2+L^2}} - \frac{1}{-L+\sqrt{y^2+L^2}}\right)\hat{y}$$

$$\Rightarrow \boxed{\vec{E}(y) = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, 2L}{y\sqrt{y^2+L^2}}\hat{y} = \frac{1}{4\pi\epsilon_0}\frac{Q}{y\sqrt{y^2+L^2}}\hat{y}}$$

As before

Check limits of $V(y)$

$y \to \infty$, i.e. $y \gg L$, i.e. $\frac{L}{y} \ll 1$

$$\Rightarrow V(y) = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L + y\left(1+\frac{L^2}{y^2}\right)^{1/2}}{-L + y\left(1+\frac{L^2}{y^2}\right)^{1/2}}\right]$$

$$\simeq \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L+y}{-L+y}\right] = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{1+\frac{L}{y}}{1-L/y}\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(1+\frac{L}{y}\right) - \ln\left(1-\frac{L}{y}\right)\right]$$

(Next Page)

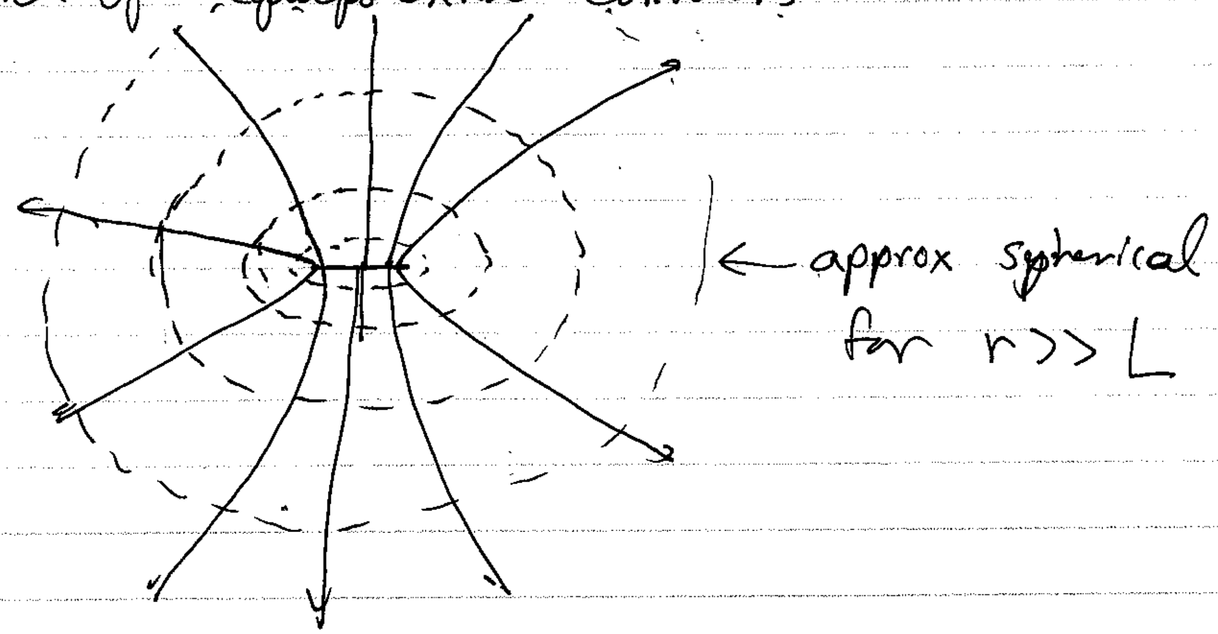
Aside: $\ln(1+\epsilon) \approx \epsilon$ for $\epsilon \ll 1$

$$\Rightarrow \text{for } y \gg L, \quad V(y) \approx \frac{\lambda}{4\pi\epsilon_0}\left[\frac{L}{y} - \left(-\frac{L}{y}\right)\right]$$

$$\Rightarrow V(y) \approx \frac{2L\lambda}{4\pi\epsilon_0\, y} = \frac{1}{4\pi\epsilon_0}\frac{Q}{y}$$

Potential of a point charge

Sketch of equipotential contours



In this example, calculating $\vec{E}$ and V on y is about the same. In more complex cases, must easier to calculate V first, the take gradient.