UNM Physics 405: Lecture 10

Summary of Electrostatics

$$\boxed{\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \rho/\epsilon_0 \\ \vec{\nabla}\times\vec{E} &= 0\end{aligned}} \Bigg\} \Rightarrow \vec{E}(\vec{r}) = \int d^3r' \, \frac{1}{4\pi\epsilon_0}\, \rho(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

Equivalent formulation

$$\vec{\nabla}\times\vec{E} = 0 \implies \vec{E} = -\vec{\nabla}V$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \implies -\vec{\nabla}\cdot(\vec{\nabla}V) = -\nabla^2 V = \rho/\epsilon_0$$

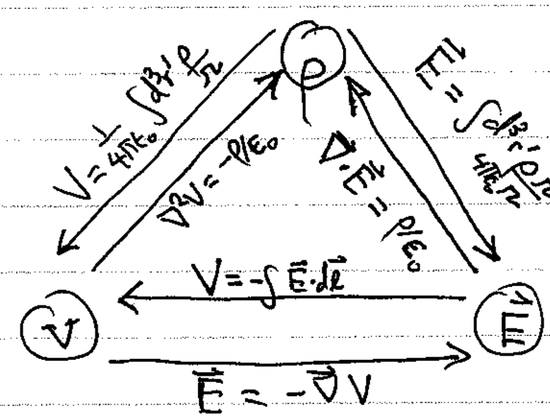
(Recall $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$: Laplacian)

$$\underline{\nabla^2 V = -\rho/\epsilon_0} \quad : \text{Poisson's Equation}$$

Solution: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\, d^3r'}{|\vec{r}-\vec{r}\,'|}$

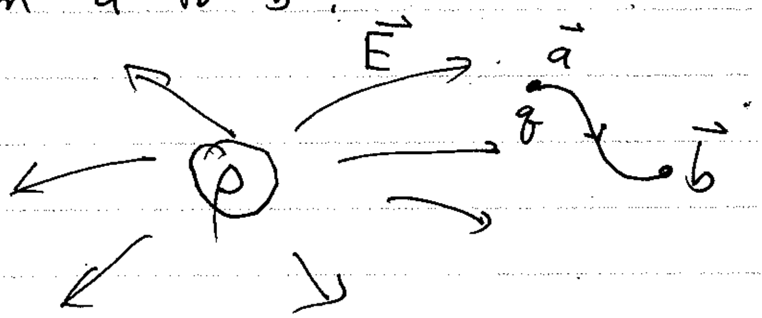
$$\vec{E} = -\vec{\nabla}V \implies \text{Coulomb's Law}$$

In charge-free region of space $\rho = 0 \Rightarrow \underline{\nabla^2 V = 0}$ Laplace's Eq



Work and Energy in Electrostatics

How much work is done on a charge by an <u>external</u> field $\vec{E}$ in moving the charge from $\vec{a}$ to $\vec{b}$?



$$W = \int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l} = q \int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = qV(\vec{a}) - qV(\vec{b})$$

The work done ~~is the~~ comes from the <u>potential energy</u> $\quad W_{\vec{a}\to\vec{b}} = -$

$$\Rightarrow \boxed{U(\vec{r}) = qV(\vec{r})}$$ is the <u>potential energy</u> of a charge q in an <u>external</u> electric ~~poten~~ field with electrostatic potential $V(\vec{r})$ relative to "ground".

Units $\quad [U] \equiv$ Joule

$$\Rightarrow [V] \equiv \text{Volt} = \frac{\text{Joule}}{\text{Coulomb}}$$

Note $eV \equiv$ "electron volt" $= 1.6\times10^{-19}$ Joules

The "negative sign" in the definition

$$V = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l} \quad \text{comes from}$$

the connection to potential energy; potential energy decreases as the particle moves according to external force

Electrostatic fields are thus "conservative"

Defining energy $\mathcal{E}$ of particle

$$\mathcal{E} = \frac{1}{2}mv^2 + U(\vec{r})$$

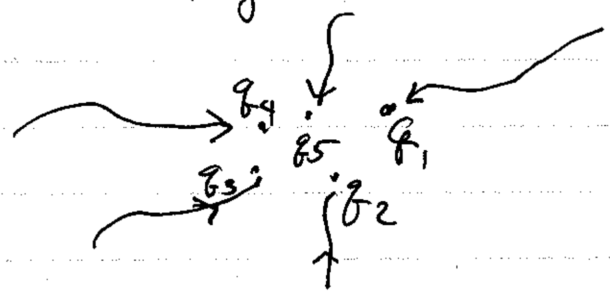
$$\Rightarrow \frac{d\mathcal{E}}{dt} = \vec{v}\cdot\underbrace{m\frac{d\vec{v}}{dt}}_{q\vec{E}} + \underbrace{\frac{dU}{dt}}_{\dot{\vec{r}}\cdot\underbrace{\vec{\nabla}U}_{+q\vec{\nabla}V = -q\vec{E}}}$$

$$\Rightarrow \frac{d\mathcal{E}}{dt} = 0 \Rightarrow \underline{\mathcal{E} \text{ is conserved}}$$

Change in potential compensated by a change in kinetic energy

Energy necessary to assemble a charge distribution

Consider a scenario in which we assemble a collection of point charges $\{q_i\}$ at positions $\{\vec{r}_i\}$ by bringing them in, point by point, from "infinity"



- Work to bring q_1 to $\vec{r}_1$ $\quad W_1 = 0$ (No external force)
- Work to bring q_2 to $\vec{r}_2$ = Work necessary to "oppose" force of charge q_1 = − Work done by 1 of 2 in bringing q_2 from ∞ to $\vec{r}_2$ = Potential energy of q_2 @ $\vec{r}_1$

$$\Rightarrow \quad W_2 = q_2 \left[V_{\text{from } 1}(\vec{r}_2) \right]$$

$$= q_2 \left(\frac{1}{4\pi\epsilon_0} \frac{q_1}{|\vec{r}_2 - \vec{r}_1|} \right)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|}$$

- Work to bring q_3 to $\vec{r}_3$ = -(Work done by 1 and 2 on q_3) = Potential energy of q_3 in the field of q_1 and q_3.

$$\Rightarrow W_3 = q_3 V_{\text{from 1 and 2}}(\vec{r}_3)$$

$$= q_3 \left(\frac{1}{4\pi\epsilon_0} \left(\frac{q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{q_2}{|\vec{r}_3 - \vec{r}_2|} \right) \right)$$

$\vdots$ etc.

$$\Rightarrow W_{\substack{\text{total to assemble}\\ \text{charge distribution}}} = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|} + \frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|} + \cdots \right.$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \left(\frac{1}{2}\right) \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

to avoid double counting

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left(\frac{1}{4\pi\epsilon_0} \sum_{j\neq i}^{n} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} \right)}_{V(\vec{r}_i)}$$

$$\Rightarrow \boxed{W_{\substack{\text{to assemble}\\ \text{charge distribution}}} = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)}$$

"self potential"

So given ~~the~~ a charge distribution of points, the work ~~the~~ needed to assemble it is

$$W_{assemble} = \frac{1}{2}\sum_{i=1}^{n} q_i \, V_{self}(\vec{r}_i)$$

where $V(\vec{r}_i)$ is the "self" potential of the distribution.

This should be contrasted with the work done by an external ~~potent~~ electrostatic force described by an external potential on collection

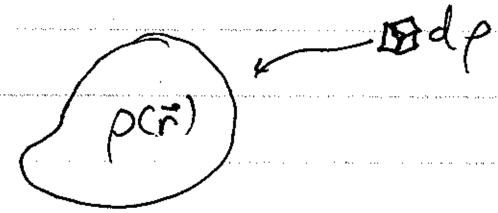
$$W_{external} = \sum_i q_i \, V_{external}(\vec{r}_i) \quad (\text{factor of } \tfrac{1}{2})$$

The work done to assemble a distribution is the energy we would get back if we "let the charges go" and allowed them to move of to ∞

$$\Rightarrow \boxed{W_{assemble} = \text{Potential energy stored in the charge distribution}}$$

$$\Rightarrow \boxed{U_{stored} = \frac{1}{2}\sum_i q_i \, V_{self}(\vec{r}_i)}$$

For a continuous charge distribution



$$\boxed{U_{\text{stored}} = \frac{1}{2}\int d^3r \; \rho(\vec{r})\, V(\vec{r})}$$

<u>Aside:</u> $\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E} \Rightarrow U = \frac{\epsilon_0}{2}\int d^3r \,(\vec{\nabla}\cdot\vec{E})\, V(\vec{r})$

<u>Aside:</u> $\vec{\nabla}\cdot\left(V(\vec{r})\,\vec{E}(\vec{r})\right) = \vec{E}\cdot\vec{\nabla}V + V(\vec{\nabla}\cdot\vec{E})$

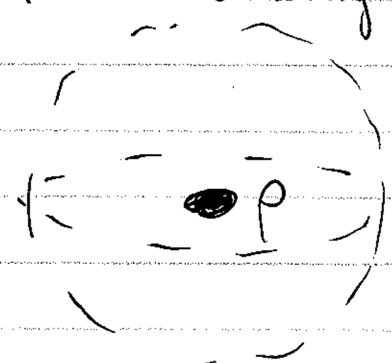
$$= -|\vec{E}|^2 + V(\vec{r})\,(\vec{\nabla}\cdot\vec{E})$$

$$\Rightarrow U = \frac{\epsilon_0}{2}\int_{\text{all space}} d^3r \; \vec{\nabla}\cdot\left(V(\vec{r})\,\vec{E}(\vec{r})\right) + \frac{\epsilon_0}{2}\int_{\text{all space}} |\vec{E}|^2 d^3r$$

(fancy integration by parts)

This expression assumes charges confined to finite region of space.

We can do integration over a finite volume and take bounding surface to infinity



← Surface bounding region of integration S

Take $S \to \infty$ at end

By the divergence theorem:

$$\int d^3r\, \vec{\nabla}\cdot(\vec{E}V) = \oint_S d\vec{a}\cdot(\vec{E}V) \to 0$$

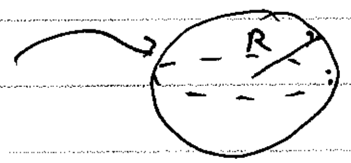
as $S\to\infty$ since field are zero @ ∞

$$\therefore \boxed{U_{\substack{\text{stored in}\\ \text{charge distribution}}} = \epsilon_0 \int_{\substack{\text{all}\\ \text{space}}} d^3r\, |\vec{E}(\vec{r})|^2}$$

<u>New interpretation</u>: The work we do to assemble a charge distribution is <u>stored</u> in the <u>electric field</u>. The electric field has energy! ~~It~~ It is a physical entity. At this point this is just bookkeeping; we can equally well consider the energy as potential energy of the charge distribution. However, we will see in 406 the ~~thes~~ field can <u>radiate</u> away from the charges, "flying free" and carrying its energy with it.

Example: Electrostatic potential energy of a spherical shell of charge, radius R, total charge Q

$$\sigma = \frac{Q}{4\pi R^2}$$



Two methods:

(1) $$U = \frac{1}{2}\int d^3r\, \rho V = \frac{1}{2}\int da\, \sigma V$$

$$= \frac{A}{2}\sigma V(R) \quad \text{since } V \text{ constant on surface}$$

$$= \left(\frac{Q}{2}\right)\frac{Q}{4\pi\epsilon_0 R} = \frac{Q^2}{8\pi\epsilon_0 R}$$

(2) $$U = \frac{\epsilon_0}{2}\int d^3r\, |\vec{E}(\vec{r})|^2 \qquad \vec{E}(\vec{r}) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & r > R \\ 0 & r < R \end{cases}$$

$$\Rightarrow U = \frac{\epsilon_0}{2}\int_R^\infty 4\pi r^2\, dr\, \frac{Q^2}{(4\pi\epsilon_0 r^2)^2}$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left(\int_R^\infty \frac{dr}{r^2}\right)$$

$$= \frac{Q^2}{8\pi\epsilon_0 R} \quad \text{(as before)}$$