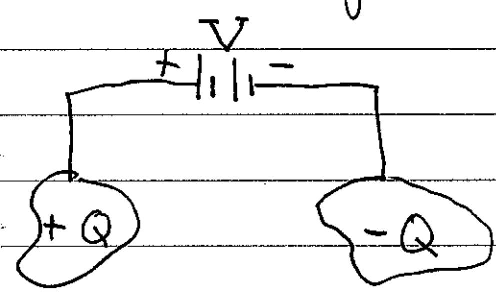
UNM Physics 405: Lecture 12

Capacitance

An important concept in electrostatics is the capacity for a set of conductors to hold charge, or the "capacitance".

Consider two conductors held at some potential difference by an external source (battery)



A total surface charge +Q will develop on one conductor and −Q on the other

The surface charge is proportional to V

$\Rightarrow$ Define $C \equiv \frac{Q}{V}$ = Capacitance

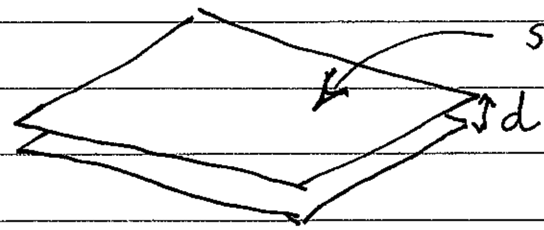
C is purely geometrical

To calculate C we place charge $+Q$, $-Q$ on two conductors and determine the potential difference between them.

Unit of capacitance $[C] \equiv$ Farad $= \frac{\text{Coul}}{\text{Volt}}$

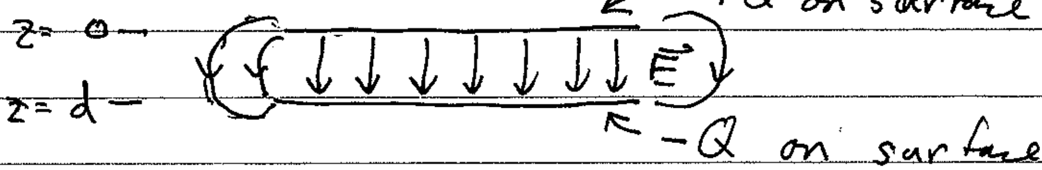
1 Farad is HUGE

Canonical example: the parallel plate capacitor



We will approximate the plates a infinite in extent
valid when $d \ll$ width and $d \ll$ length

"Edge on" view:



Ignoring "fringing field" $|\vec{E}| = \frac{Q}{\epsilon_0 A}$ (from Gauss' Law)

Potential Difference $V = -\int_d^0 E(dz) = Ed$

$$\Rightarrow \quad V = \frac{Qd}{A\epsilon_0}, \qquad C = \frac{Q}{V}$$

$$\Rightarrow \boxed{C = \epsilon_0 \frac{A}{d}}$$ Parallel Plate Capacitor

General forms for two conductor geometry

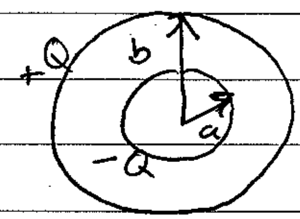
$$\boxed{C \sim \epsilon_0 \frac{\text{Surface area between}}{\text{distance between conductors}}}$$

Units: $\epsilon_0 = 8.85 \times 10^{-12} \left(\frac{\text{Coul}^2}{N\,m^2} = \frac{\text{Coul}}{\frac{\text{Joul}}{\text{coul}}\, m} = F/m \right)$

$\Rightarrow \boxed{\epsilon_0 = 8.85\ pF/m}$ pico farad

Example:

Capacitance of concentric spheres



$$E = \begin{cases} 0 & r<a \text{ or } r>b \\ -\dfrac{Q}{4\pi\epsilon_0 r^2} & a<r<b \end{cases}$$

$$V_{+-} = -\int_{\vec{r}_-}^{\vec{r}_+} \vec{E}\cdot d\vec{l} = -\int_a^b E(r)\,dr = \frac{Q}{4\pi\epsilon_0}\int_a^b \frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)_a^b = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\Rightarrow \boxed{V_{+-} = \frac{Q}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right)}$$ ~~Exact~~ result

Consider the limit as the spacing between the spheres becomes small. We expect this to approach the geometry of a parallel plate (flat earth society).

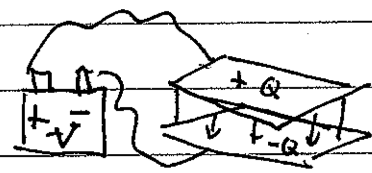
Exact $$\boxed{C = \frac{Q}{V_{+-}} = 4\pi\epsilon_0\frac{ab}{b-a}}$$ Concentric spheres

$\begin{pmatrix}\text{Limit}\\ b=a+d\\ d\to 0\end{pmatrix}$ $$C = 4\pi\epsilon_0\frac{a(a+d)}{(a+d)-a} = 4\pi\epsilon_0\left(\frac{a^2+\cancel{ad}^{\,0}}{d}\right)$$

$$\cong \epsilon_0(4\pi d^2)\frac{1}{d} = \epsilon_0\frac{A}{d}\checkmark$$

Energy stored in capacitor:

An external source (battery) must do work to move charge from one conductor onto the other (the energy stored in the charge distribution)



To move charge dq against the field established by charges $\pm q$ already on capacitor requires work dW

$$dW = dq\,\underbrace{\left(\frac{q}{C}\right)}_{\text{potential difference}} \implies \text{Total Work } W = \int dW$$

$$\boxed{W = \int_0^Q dq\left(\frac{q}{C}\right) = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2}$$

Equivalent to think about energy stored in field

$$W = U = \frac{\epsilon_0}{2}\int E^2\, d^3r$$

Example: Parallel Plate Capacitor

$$E \cong \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0} \quad \text{uniform inside, zero out}$$

$$\implies U \cong \frac{\epsilon_0}{2}E^2\underbrace{(Ad)}_{\text{volume where } E\neq 0} = \frac{1}{2}Q^2\left(\frac{d}{A\epsilon_0}\right) = \frac{1}{2}\frac{Q^2}{C} \checkmark$$

Example, concentric spheres

$$U = \frac{\epsilon_0}{2}\int |\vec{E}|^2 d^3r = \frac{\epsilon_0}{2}\int_a^b 4\pi r^2 dr \, \frac{Q^2}{16\pi^2\epsilon_0^2 r^4}$$

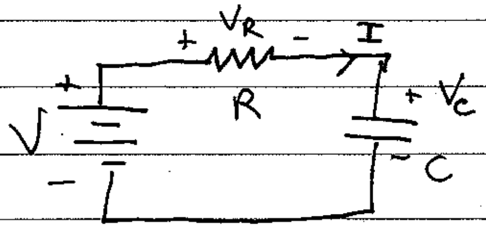
$$= \frac{Q^2}{8\pi\epsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \checkmark$$

Charging up capacitors

Let us take a small detour away from electrostatics and consider the flow of charge (current) in a circuit.

"RC circuit"



$$\oint \vec{E}\cdot d\vec{\ell} = 0 \quad \text{(Kircoff's Law)}$$

$$\Rightarrow V = V_R + V_C$$

$$V_R = I = \left(\frac{dQ}{dt}\right) R \quad \left(\text{Ohm's Law}\right)$$

$$V_C = Q/C$$

$$\Rightarrow R\frac{dQ}{dt} + \frac{Q}{C} = V \Rightarrow \boxed{\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{V}{R}}$$

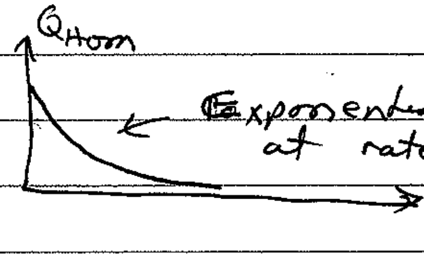
First order diff' eq

Solution: $Q(t) = Q_{\text{Homogenous}}(t) + Q_{\text{Particular}}(t)$

Homogeneous Solution (set $V = 0$)

$$\frac{dQ_{Hom}}{dt} + \frac{1}{RC} Q_{Hom} = 0 \Rightarrow Q_{Hom}(t) = K e^{-t/RC}$$

"RC time constant"



Exponential decay at rate $\frac{1}{RC}$

Particular solution = Steady state $\frac{dQ_{Part}}{dt} = 0$

$$\Rightarrow \frac{1}{RC} Q_{Part} = V/R \Rightarrow Q_{Part} = CV$$

$$\Rightarrow Q(t) = K e^{-t/RC} + CV$$

Find K from initial condition $Q(0)$

$$\Rightarrow Q(0) = K + CV \Rightarrow K = Q(0) - CV$$

$$\Rightarrow \boxed{Q(t) = \underbrace{Q(0)\, e^{-t/RC}}_{\text{Decay of initial state}} + \underbrace{CV\left(1 - e^{-t/RC}\right)}_{\text{Charge up to final state}}}$$