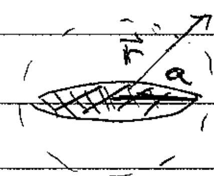
Physics 405 - Lecture 17

Multipole Expansion (Intro)

We saw at the end of lecture 16, that for an azimuthally symmetric charge distrubution (e.g. a circular disk), the potential outside $r > a$



has the form

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$= \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots$$

For the case of the disk

$$B_\ell = 0 \qquad \ell \text{ odd}$$

$$B_0 = \frac{Q}{4\pi\epsilon_0}, \quad B_2 = \frac{1}{4\pi\epsilon_0}\left(-\frac{1}{4}Qa^2\right), \quad \ldots$$

As $r \to \infty$ $V(r,\theta) \to \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$, the field of of point charge

The term $\propto \frac{1}{r^3}$ is the next correction describes the finite extent of the distribution

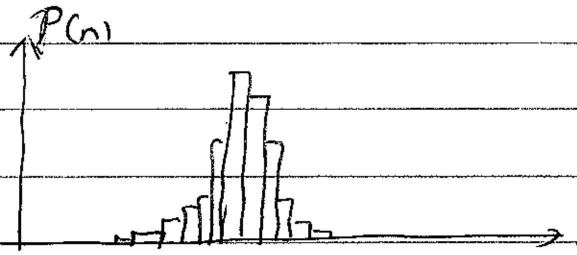
The coefficients $4\pi\epsilon_0 B_\ell$ are examples of "multipole moments" of the charge distribution

$\ell=0 \quad 4\pi\epsilon_0 B_0 = Q$ "monopole moment"

$\ell=2 \quad 4\pi\epsilon_0 B_2 = \frac{1}{4}Qa^2$ "quadrupole moment"

The notion of "moments" of a distribution is an important concept, typically discussed in statistics

Example: Number of particles at discrete positions, n



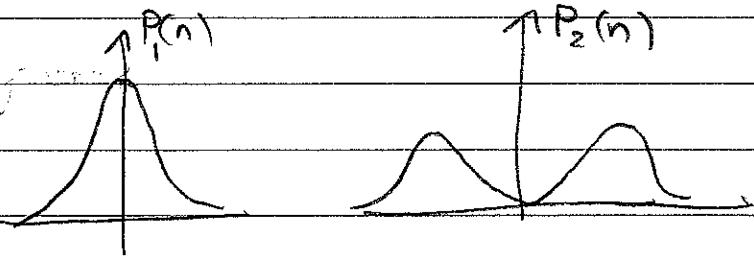
This distribution is characterized by its "moments"

$$\sum_n P(n) = N_{tot} = \text{total \# of particles}$$

$$\sum n P(n) = N_{tot}\langle n\rangle = \text{Average position of detection} \times (\text{total \#})$$

$$\sum n^2 P(n) = N_{total}\langle n^2\rangle = \text{Average spread about origin} \times (\text{total \#})$$

Ex:



$$N^1_{tot} = N^2_{tot}$$

$$\langle n\rangle_1 = \langle n\rangle_2$$

But $\langle n^2\rangle_1 \neq \langle n^2\rangle_2$

In the context of charge distributions $\rho(\vec{r})$ this in complicated by two factors

(1) We are dealing with three dimensions
(2) Charges come in positive and negative

- Since we are dealing with 3D their are a huge variety of moments

$$\text{Eg.} \quad \langle x^2 \rangle = \int d^3r \, x^2 \rho(\vec{r})$$

$$\langle yz \rangle = \int d^3r \, yz \, \rho(\vec{r})$$

So the moments are generally defined as component of a tensor

- Because we have both positive and negative charges, we can consider separate distributions

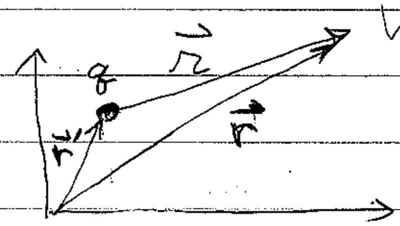
$$\rho(\vec{r}) = \underbrace{\rho_+(\vec{r})}_{\text{distribution of positive charge}} + \underbrace{\rho_-(\vec{r})}_{\text{distribution of negative charge}}$$

The moments of the charge distribution complete characterize the distribution, and thus the electrostatic field they generate. This is particularly useful when we want to know the field far from the distribution where the field falls off rapidly with distance r. Then the nature of the field is dominated by the lowest non-vanishing moments.

Basic Multipoles for Point Charges

To better understand the nature of the moments, consider the characteristic potentials and fields associated with some basic collections of point charges.

(i) A single point charge: Monopole

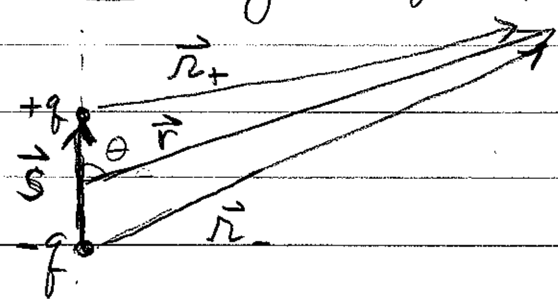


$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} \cong \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

when $|\vec{r}| \gg |\vec{r}'|$

"Monopole moment" q = total charge

(ii) Two oppositely charged point: Dipole

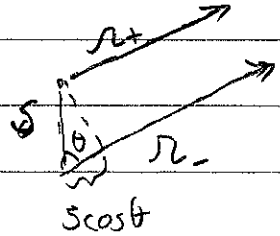


Total charge = 0

$\Rightarrow$ No "monopole pole"

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{\mathscr{r}_- - \mathscr{r}_+}{\mathscr{r}_+\mathscr{r}_-}\right)q$$

For $|\vec{r}| \gg |\vec{s}|$



$\mathscr{r}_+ \approx \mathscr{r}_- \approx r$

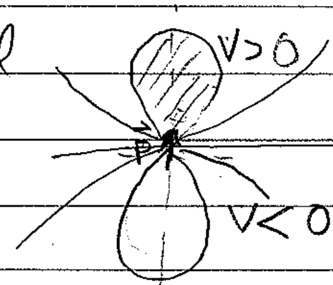
to first order $\mathscr{r}_+ - \mathscr{r}_- \approx s\cos\theta$

$$\Rightarrow V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\frac{qs\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

$\vec{p} \equiv q\vec{s}$ Electric dipole moment (vector)

Dipole potential $\sim \frac{1}{r^2}$ for $r \gg s$

Dipole equipotential



"Dipole Pattern"

Electric field

$$\vec{E}(\vec{r}) = -\vec{\nabla} V = -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \frac{\vec{p}\cdot\hat{r}}{r^2} = -\frac{1}{4\pi\epsilon_0} \vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

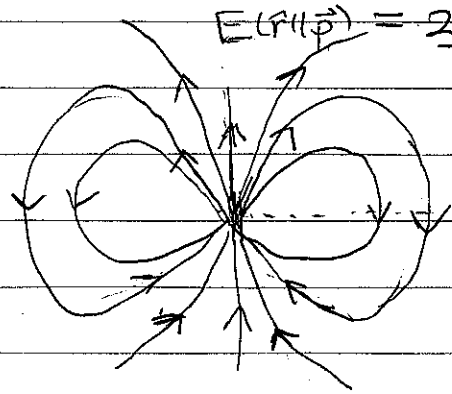
$$= -\frac{1}{4\pi\epsilon_0}\left[(\vec{p}\cdot\vec{r})\vec{\nabla}\frac{1}{r^3} + \frac{1}{r^3}(\vec{p}\cdot\vec{\nabla})\vec{r}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{3(\vec{p}\cdot\vec{r})\hat{r}}{r^4} - \frac{\vec{p}}{r^3}\right)$$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)}{r^3}}$$

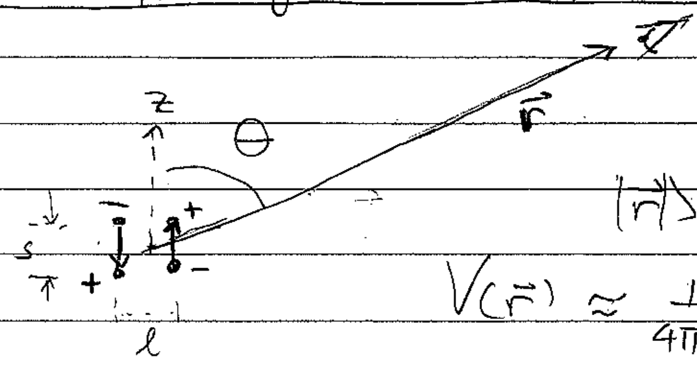
Dipole Field $\sim \frac{1}{r^3}$

$E(\hat{r} \parallel \vec{p}) = \frac{2\vec{p}}{r^3}$



$E(\hat{r} \perp \vec{p}) = -\frac{\vec{p}}{r^3}$

(iii) Two Oppositely Oriented Dipoles: Quadrupole



$|\vec{r}| \gg s, \ell$

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left(\frac{qd\cos\theta}{r_+^2} - \frac{qd\cos\theta}{r_-^2}\right)$$

$$\Rightarrow V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{qd\cos\theta\,(r_-^2 - r_+^2)}{r_+^2 r_-^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{qd\cos\theta\,(r_- + r_+)\overbrace{(r_- - r_+)}^{\approx 2s\sin\theta}}{r_+^2 r_-^2}$$

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{(2qds\sin\theta\cos\theta)}{r^3}$$ ← One component of the "quadrupole tensor")

Equipotential contours