Physics 405 - Lecture 18

Multipole Expansion in Electrostatics

We saw in lecture 17 how collections of point charges lead to characteristic potentials far from the charges

$$V(r) = \frac{q}{4\pi\epsilon_0 r} \quad \text{(monopole)}$$

$$V(r) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} \quad \text{(dipole)}$$

$$V(r) = \frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{4\pi\epsilon_0 r^3} \quad \text{(Quadrapole)}$$

$$\vdots$$

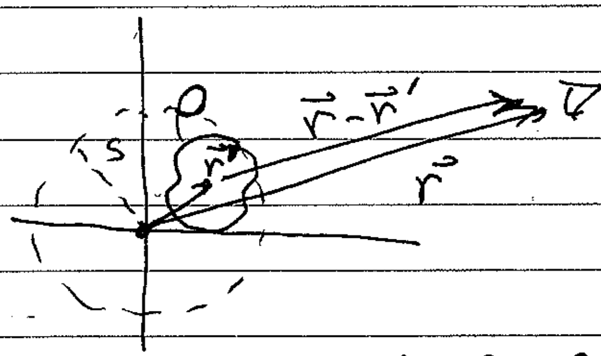
We would like to obtain an expression for multipole moments (generally tensors) for an arbitrary (localized) charge distribution this may be too ambitious - we will eventually restrict our attention to azimuthally symmetric distributions

Return to the general expression for the potential

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

with ground @ infinity

We consider the charge distribution localized with a given radius



We seek the potential far from the distribution $|\vec{r}| >> |\vec{r}'|$.

Aside: $$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')}} = \frac{1}{\sqrt{r^2-2\vec{r}\cdot\vec{r}'+r'^2}}$$

$$= \frac{1}{r}\left(1-2\frac{\vec{r}\cdot\vec{r}'}{r^2}+\frac{r'^2}{r^2}\right)^{-1/2}$$

Consider a Taylor series expansion in $\frac{r'}{r}$

To first order $$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\left(1+\frac{\vec{r}\cdot\vec{r}'}{r^2}+\mathcal{O}\left(\left(\frac{r'}{r}\right)^2\right) + \cdots\right)$$

$$\Rightarrow V(r) = \frac{1}{4\pi\epsilon_0}\frac{\int d^3r'\,\rho(\vec{r}')}{r} + \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\int d^3r'\,\vec{r}'\rho(\vec{r}')}{r^2}$$

$$+ \cdots$$

$$\Rightarrow V(r) = \frac{1}{4\pi\epsilon_0} \frac{q_0}{r} + \frac{1}{4\pi\epsilon_0} \frac{\hat{r} \cdot \vec{p}}{r^2} + \cdots$$

where $\boxed{q_0 = \int d^3r' \, \rho(\vec{r}\,') \text{ (monopole moment)} = \text{total charge}}$

$$\boxed{\begin{aligned} \vec{p} &= \int d^3r' \, \vec{r}\,' \rho(\vec{r}\,') \quad \text{(dipole moment)} \\ &= |q_{pos}| \langle \vec{r} \rangle_{pos} - |q_{neg}| \langle \vec{r} \rangle_{neg} \end{aligned}}$$

The higher order terms in the Taylor series expansion lead the higher order moments (tensors)

Note: The lowest nonvanishing multipole moment is independent of origin, but higher order moments depend on origin.

E.g. Consider the dipole moment

$$\vec{p} = \int d^3r' \, \vec{r}\,' \rho(\vec{r}\,')$$

Change origin: $\vec{r}\,' \Rightarrow \vec{r}\,' + \vec{r}_0$

$$\Rightarrow \quad \vec{p} \to \int d^3r' \, (\vec{r}\,' + \vec{r}_0) \rho(\vec{r}\,') = \vec{p} + \vec{r}_0 q_0$$

If q_0 vanishes $\vec{p}$ is unchanged, otherwise $\vec{p}$ depends on $\vec{r}_0$

Multipole Expansion for Azimuthal Symmetry

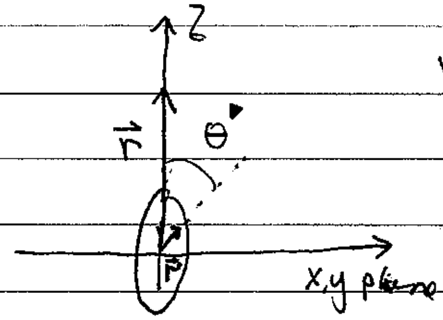
In order simplify, we will obtain the general multipole moments for a charge distribution that is azimuthally symmetric, $\rho(r,\theta)$.

We know from Lect. 17, that if we determine V on the axis of symmetry (z-axis, $\theta=0$), we can find V at all positions according to the following procedure:

$$\vec{V}(r=z, \theta=0) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{z^{\ell+1}} \quad \text{when } |z| > \text{radius of charge distribution}$$

$$\Rightarrow V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Consider then the general expression for V on the z-axis



$$V(r=z, \theta=0) = \int d^3r' \frac{\rho(\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

Now, from P.S. #7 we know

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\gamma)$$

when $r > r'$ and γ is the angle between $\vec{r}$ and $\vec{r}'$

Thus, when $\vec{r}$ is on the z-axis, $\gamma = \theta'$

$$\Rightarrow V(r=z, \theta=0) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{\int d^3r' \, r'^{\ell} \rho(r', \theta') P_\ell(\cos\theta')}{z^{\ell+1}}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{Q^{(\ell)}}{z^{\ell+1}}$$

where $\boxed{Q^{(\ell)} \equiv \int d^3r \, r^{\ell} \rho(r,\theta) P_\ell(\cos\theta) = \ell^{th} \text{ order multipole moment}}$

$$\Rightarrow \boxed{V(r,\theta) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{Q^{(\ell)}}{r^{\ell+1}} P_\ell(\cos\theta)}$$

The ℓ^{th} order multipole is determined by projection of the charge distribution on the ℓ^{th} order Legendre polynomial. Each moment, for the azimuthally symmetric distribution is characterized by one #.

$\ell=0$ $\quad Q^{(0)} = \int d^3r \, \rho(\vec{r})$ = total charge (monopole)

$\ell=1$ $\quad Q^{(1)} = \int d^3r \, r P_1(\cos\theta) \, \rho(\vec{r})$

$$= \int d^3r \, r\cos\theta \, \rho(\vec{r})$$

$$= \int d^3r \, z \, \rho(\vec{r}) = \text{z-component of dipole}$$

(dipole on z-axis) $= |\vec{p}|$

Example:

We saw that when a conducting sphere is placed in a uniform electric field, a charge density is induced on the surface $\sigma(\theta) \sim \sigma_0 \cos\theta$.

What are the multipoles?

$$Q^{(\ell)} = \int d^3r \; r^\ell \rho(r,\theta) \; P_\ell(\cos\theta)$$

$$= \int da \; R^\ell \; \sigma(\theta) \; P_\ell(\cos\theta)$$

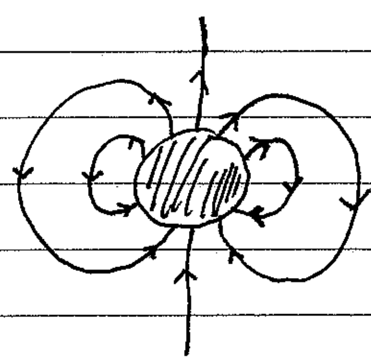
$$R^2 d\phi \, d(\cos\theta) = 2\pi R^2 \, d(\cos\theta)$$

$$= 2\pi R^2 \int d(\cos\theta) \; R^\ell \; \sigma_0 \cos\theta \; P_\ell(\cos\theta)$$

$$= 2\pi R^{\ell+2} \sigma_0 \underbrace{\int d(\cos\theta) \; \cos\theta \; P_\ell(\cos\theta)}_{\delta_{\ell 1}}$$

$\delta_{\ell 1}$ since $\cos\theta = P_1(\cos\theta)$

$\Rightarrow$ Pure dipole $\quad Q^{(1)} = 2\pi R^3 \sigma_0$

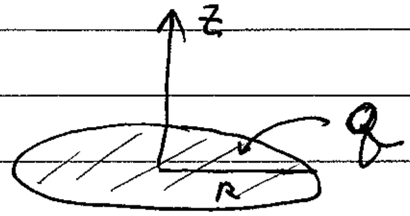


For $\sigma_0 = 3\epsilon_0 E$

$$\Rightarrow V_{\text{induced}} = \frac{1}{4\pi\epsilon_0} \frac{Q^{(1)}\cos\theta}{r^2}$$

$$= \frac{3R^3 E_0}{2} \frac{\cos\theta}{r^2}$$

Example: Disk of radius R, uniformly distributed charge q



- Origin @ center of disk
- z along symmetry axis

- Monopole moment: q
- Dipole moment: 0 $\left(\begin{array}{l}\text{average position pos. charge} \\ = \quad " \quad\quad " \quad \text{neg. charge}\end{array}\right)$
- Quadrupole moment:
  Since charge is distributed on a surface, rather than throughout a volume, we integrate over the area at $\theta = 0$

$$\Rightarrow Q^{(2)} = \int_{\text{disk}} da\, r^2 \sigma\, P_2\left(\cos\left(\theta = \tfrac{\pi}{2}\right)\right)$$

Aside: For disk $da = 2\pi r\, dr$, $P_2(\cos\frac{\pi}{2}) = P_2(0) = -\frac{1}{2}$

$$\Rightarrow Q^{(2)} = -\pi\left(\sigma = \frac{q}{\pi R^2}\right)\int_0^R r\, dr\, r^2$$

$$= -\frac{Q}{R^2}\left[\frac{r^4}{4}\right]_0^R = \boxed{-\frac{1}{4} q R^2}$$

$\Rightarrow$ To order $\left(\frac{R}{r}\right)^3$

as before

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} - \frac{1}{4}\frac{qR^2}{r^3} P_2(\cos\theta)\right]$$