Physics 405

Problem Set #1 Solutions

(1) Hill height $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$

(a) The top of the hill is the point (x,y) where $\vec{\nabla} h = 0$

$$\vec{\nabla} h = \frac{\partial h}{\partial x}\hat{x} + \frac{\partial h}{\partial y}\hat{y} = 10(2y - 6x - 18)\hat{x}$$
$$+10(2x - 8y + 28)\hat{y}$$

$\vec{\nabla} h = 0 \Rightarrow$ Both x and y components are zero

$$\Rightarrow \begin{array}{l} 10(2y - 6x - 18) = 0 \\ 10(2x - 8y + 28) = 0 \end{array} \Rightarrow \begin{bmatrix} -6 & 2 \\ 2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ -28 \end{bmatrix}$$

(matrix form)

Inverting $\Rightarrow$ $x = -2$ $y = 3$

Answer: Top of the hill 2 miles ~~[illegible]~~ west and 3 miles north of South Hadley

(b) The height of the hill is the value of h at $(x = -2,\ y = 3)$

$$\boxed{h(-2,3) = 720 \text{ feet}}$$

(c) Steepness $\equiv$ Magnitude of the gradient

$$\vec{\nabla} h(1,1) = 10(2-6-18)\hat{x} + 10(2-8+28)\hat{y} = -220\hat{x} + 220\hat{y}$$

Steepness $= |\vec{\nabla} h(1,1)| = 220\sqrt{2}$ feet/mile

direction $= \dfrac{\vec{\nabla} h}{|\vec{\nabla} h|} = -\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}$

Problem 1 d

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h[x_,y_] := 10*( 2 x y - 3 x^2 - 4 y^2 -
          18 x + 28 y +12)
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Plot3D[h[x,y],{x,-5,0},{y,0,5}]
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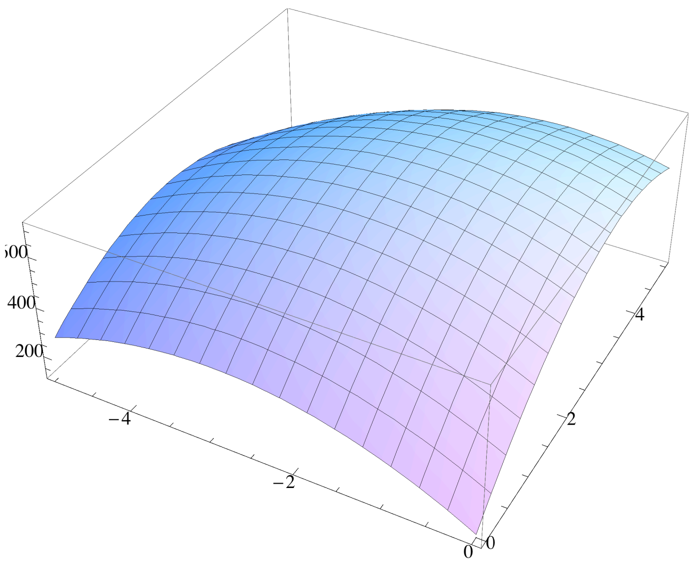


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contours = ContourPlot[h[x,y],{x,-4,0},{y,0,6}]
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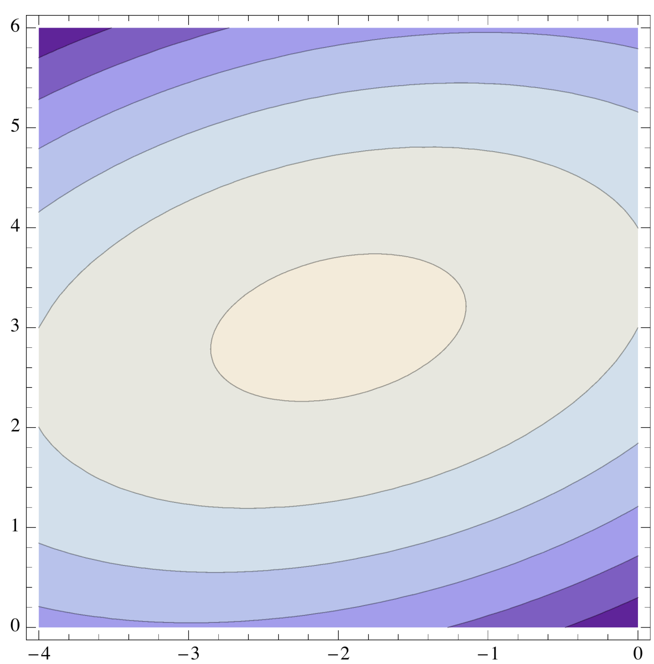


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Needs["VectorFieldPlots`"]
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fields = GradientFieldPlot[h[x, y], {x, -4, 0}, {y, 0, 6}, PlotPoints → 10]
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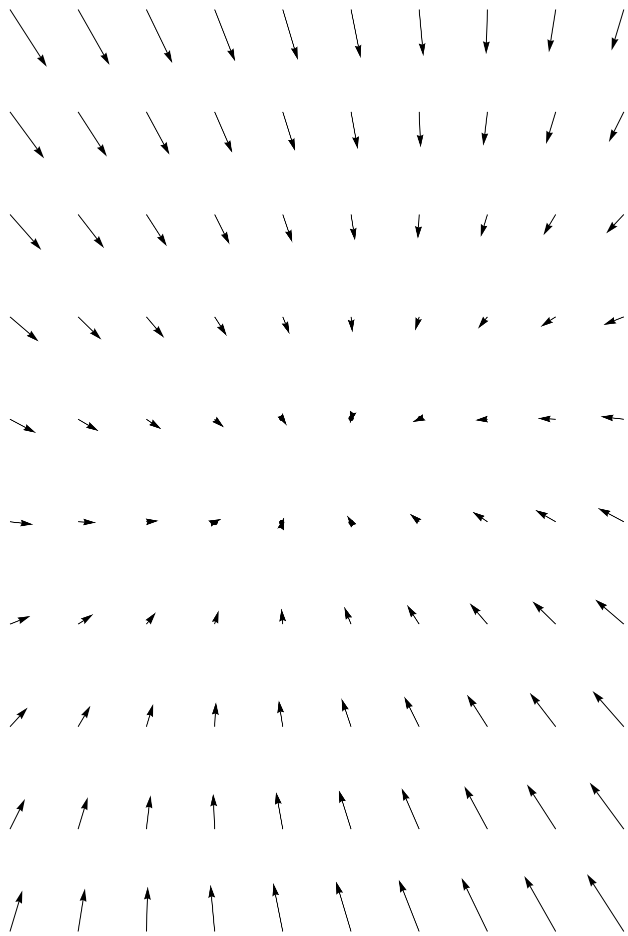

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Show[contours, fields]
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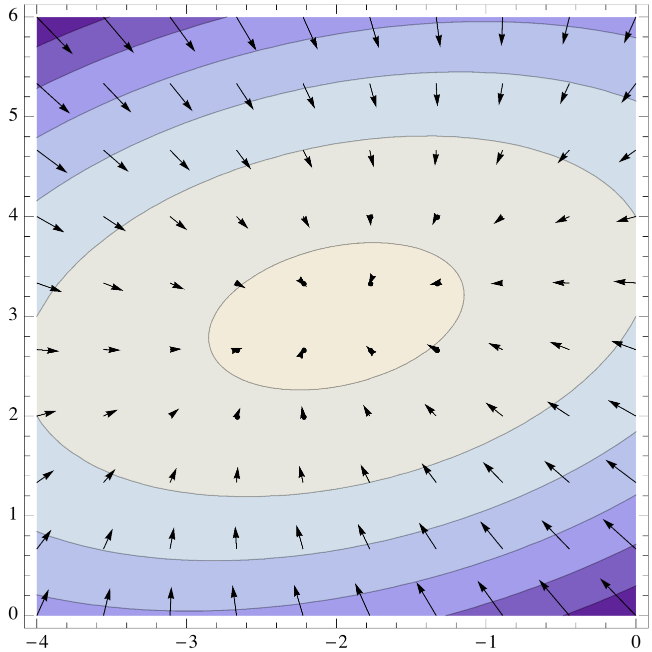


Problem 2

(a) $\vec{v} = x^2\,\hat{x} + 3xz^2\,\hat{y} - 2xz\,\hat{z}$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz)$$

$$= 2x - 2x = \boxed{0 = \vec{\nabla}\cdot\vec{v}}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = \hat{x}\left(\cancelto{0}{\frac{\partial}{\partial y}(-2xz)} - \frac{\partial}{\partial z}(3xz^2)\right)$$

$$-\hat{y}\left(\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial z}(x^2)\right)$$

$$+\hat{z}\left(\frac{\partial}{\partial x}(3xz^2) - \cancelto{0}{\frac{\partial}{\partial y}(x^2)}\right)$$

$$\boxed{\vec{\nabla}\times\vec{v} = -6xz\,\hat{x} + 2z\,\hat{y} + 3z^2\,\hat{z}}$$

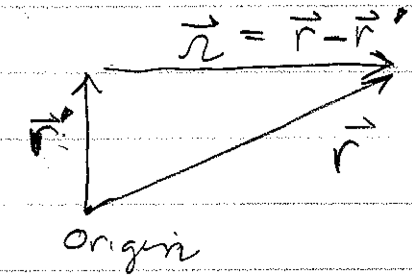
(b) $\vec{v} = \hat{x}\cos(kz - \omega t)$ (Here t is a parameter)

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}\left(\cos(kz - \omega t)\right) = \boxed{0}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(kz-\omega t) & 0 & 0 \end{vmatrix} = -\hat{y}\left(-\frac{\partial}{\partial z}\cos(kz-\omega t)\right)$$

$$= \boxed{-\hat{y}\,k\sin(kz-\omega t)}$$

Problem 3: Practice with Gradient (Fixed vs. Variable)
Griffiths 1.13 (Page 15)



$\vec{r}\,'$ = fixed vector

$\vec{r}$ = variable position

$\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}\,'$

$$\Rightarrow \mathscr{r} = |\vec{r} - \vec{r}\,'| = \sqrt{\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}} = \sqrt{(\vec{r} - \vec{r}\,') \cdot (\vec{r} - \vec{r}\,')}$$

$$= \sqrt{r^2 - 2\vec{r} \cdot \vec{r}\,' + r'^2}$$

(a) $\vec{\nabla} \mathscr{r}^2 = 2\mathscr{r} \vec{\nabla} \mathscr{r}$ (By product rule)

Aside: $\vec{\nabla} \mathscr{r} = \vec{\nabla} \left[r^2 - 2\vec{r} \cdot \vec{r}\,' + r'^2 \right]^{1/2}$

$$= \frac{1}{2} \frac{1}{\left[r^2 - 2\vec{r} \cdot \vec{r}\,' + r'^2 \right]^{1/2}} \vec{\nabla} \left(r^2 - 2\vec{r} \cdot \vec{r}\,' + r'^2 \right)$$

Double aside:

$$\vec{\nabla} r^2 = 2r \vec{\nabla} r = 2r\hat{r} = 2\vec{r}$$

$$\vec{\nabla} 2\vec{r} \cdot \vec{r}\,' = (2\vec{r}\,' \cdot \vec{\nabla})\vec{r} = 2\vec{r}\,'$$

$$\vec{\nabla} r'^2 = 0$$ since $\vec{r}\,'$ is fixed

$$\Rightarrow \vec{\nabla} \mathscr{r} = \frac{\vec{r} - \vec{r}\,'}{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \hat{\mathscr{r}}$$

$$\Rightarrow \boxed{\vec{\nabla} \mathscr{r} = 2\mathscr{r} \hat{\mathscr{r}} = 2\vec{\mathscr{r}}}$$

(b) $\vec{\nabla}\frac{1}{r} = \vec{\nabla} r^{-1} = -\frac{1}{r^2}\vec{\nabla} r$

$$\Rightarrow \boxed{\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}}$$

(c) General formula for

$$\vec{\nabla} r^n = n r^{n-1}\vec{\nabla} r = \boxed{n r^{n-1}\hat{r}}$$

$$\boxed{= n r^{n-2}\vec{r}}$$

We will see this type of formula many times when we consider the fields generated by sources (e.g. $\vec{E}$-field generated by charges)