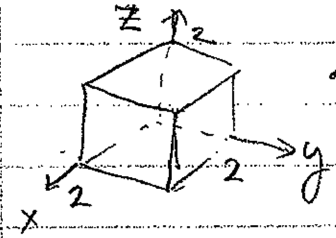
Physics 405

P.S #2 Solutions

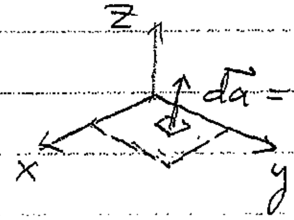
Problem 1: Notion of Flux

Griffiths: 1.29 (Page 28)



← Surface of cube

We want to consider flux through bottom surface in $+\hat{z}$ direction



$d\vec{a} = \hat{z}\,dx\,dy$ $\quad \vec{V}\cdot d\vec{a}\Big|_{\text{surface}} = V_z(x,y,0)\,dx\,dy$

(a) $\Rightarrow \int_{\text{surface}} d\vec{a}\cdot\vec{V} = \int_0^2 dx \int_0^2 dy\; y(-3) = -3\,\frac{x^2}{2}\Big|_0^2 \frac{y^2}{2}\Big|_0^2$

$= -3\left(\frac{4}{2}\right)\left(\frac{4}{2}\right) = \boxed{-12}$

(b) For the closed surface, the flux out of the bottom in the negative $\hat{z}$ direction

$\Rightarrow$ Flux out through bottom $= +12$

$\Rightarrow$ Total flux out of all faces of the cube

$= +12 + 20$ ← From Ex 1.7 of G, page 26-27

$\Rightarrow$ $\boxed{\text{Total flux of of cube} = 32}$

As a check we should have

$$\oint_{\substack{\text{surface} \\ \text{of cube}}} \vec{V} \cdot d\vec{a} = \int_{\text{Volume of cube}} (\vec{\nabla} \cdot \vec{V}) \, dx\,dy\,dz$$

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(x+2) + \frac{\partial}{\partial z}\left(y(z^2-3)\right)$$

$$= 2z + 2zy = 2z(y+1)$$

$$\Rightarrow \int_0^2 dx \int_0^2 dy \int_0^2 dz \; 2z(y+1) = \int_0^2 dx \int_0^2 (y+1)\,dy \int_0^2 dz \, 2z$$

$$= \left(x\Big|_0^2\right)\left(\frac{y^2}{2}+y\right)\Big|_0^2 \left(z^2\right)\Big|_0^2 = 2\left(\frac{4}{2}+2\right)(4) = \boxed{32}$$

Yes!

Finally, Griffiths asks, "Does the surface integral depend only on the boundary line of box"

— Well.... yes and no.

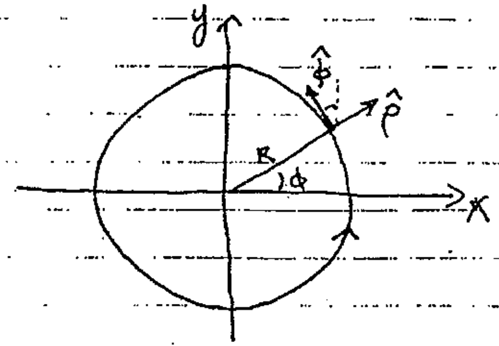
<u>We know</u> $\displaystyle\int_{\text{face}} \vec{V} \cdot d\vec{a} = \oint_{\text{boundary lines}} (\vec{\nabla} \times \vec{V}) \cdot d\vec{l}$ (Stokes' theorem)

This means the flux integral depends on the boundary line <u>and</u> the <u>curl</u> on the boundary.

(2) Check Stokes' Theorem for the vector field

$$\vec{v} = ay\,\hat{x} + bx\,\hat{y} \quad (a \text{ and } b \text{ constants})$$

and the circular path of radius R, centered at the origin



We must choose an orientation for the path (say counter-clockwise)

For this path $d\vec{l} = R\,d\phi\,\hat{\phi}$ (in polar coordinates)

where $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$, $\cos\phi = \frac{x}{R}$, $\sin\phi = \frac{y}{R}$

$$\Rightarrow \quad \vec{v}\cdot\frac{d\vec{l}}{d\phi} = Ray\,\hat{x}\cdot\hat{\phi} + Rby\,\hat{y}\cdot\hat{\phi} = Ray(-\sin\phi) + Rby(\cos\phi)$$

$$= R^2(-a\sin^2\phi + b\cos^2\phi)$$

$$= \left(\frac{b-a}{2}\right)R^2 + \left(\frac{b+a}{2}\right)R^2\cos 2\phi \quad \left(\begin{array}{c}\text{Using} \\ \sin^2\phi = \frac{1-\cos 2\phi}{2} \\ \cos^2\phi = \frac{1+\cos 2\phi}{2}\end{array}\right)$$

$$\therefore \oint_C \vec{v}\cdot d\vec{l} = \int_0^{2\pi} \vec{v}\cdot\hat{\phi}\,R\,d\phi = \int_0^{2\pi} d\phi \left[\left(\frac{b-a}{2}\right)R^2 + \left(\frac{b+a}{2}\right)R^2\cos 2\phi\right]$$

$$= \left(\frac{b-a}{2}\right)R^2 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} + \left(\frac{b+a}{2}\right)R^2 \underbrace{\int_0^{2\pi} d\phi\,\cos 2\phi}_{\sin 2\phi\,\big|_0^{2\pi} = 0}$$

$$\Rightarrow \boxed{\oint_C \vec{v}\cdot d\vec{l} = (b-a)\pi R^2}$$

✓

(2) continued

Now we want to prove $\oint_C \vec{v}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{A}$

where S is a surface whose boundary is C and the orientation of the surface is determined by the right hand rule.

Let's choose S to be the disk in the x-y plane. Since C is counter-clockwise, the normal to S is $\hat{z}$

Now $$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix} = \hat{z}(b-a)$$

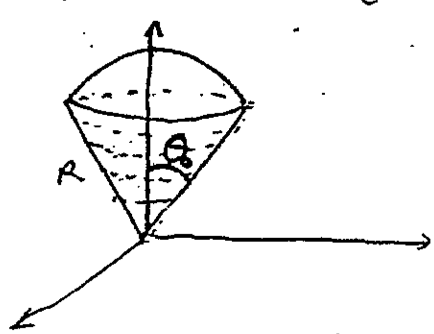
$$\therefore \int_S \vec{\nabla}\times\vec{v}\cdot d\vec{A} = \int_{Disk} \vec{\nabla}\times\vec{v}\cdot\hat{z}\, dA = (b-a)\int_{Disk} dA$$

$$= (b-a)\pi R^2 \checkmark$$

(3) Check the divergence theorem for

$$\vec{v} = r^2\sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r\tan\theta\,\hat{\phi}$$

and the "ice-cream cone"



$\theta_0 = 30° = \frac{\pi}{6}$ in radians

(Next Page)

First consider surface integral

$$\oint_S \vec{v}\cdot d\vec{A} = \int_{S_1} \vec{v}\cdot d\vec{A} + \int_{S_2} \vec{v}\cdot d\vec{A}$$

where I have broken the surface into two pieces ...

$S_1 \equiv$ surface of "ice-cream" $\quad S_2 =$ surface of "cone"

S_1

On S_1, r is constant $= R$

$$\Rightarrow |d\vec{A}| = ds_\theta(R)\, ds_\phi(R) = R^2 \sin\theta\, d\theta\, d\phi$$

the direction normal to S is $\hat{r}$

$$\Rightarrow \int_{S_1} \vec{v}\cdot d\vec{A} = \int \vec{v}\cdot\hat{r}\, dA$$

$$= \int_{S_1} v_r(R,\theta,\phi)\, R^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/6} d\theta\; R^4 \sin^2\theta$$

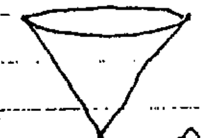
$$= 2\pi R^4 \int_0^{\pi/6} \sin^2\theta\, d\theta$$

$$= 2\pi R^4 \int_0^{\pi/6} \frac{(1-\cos 2\theta)}{2}\, d\theta$$

$$= \pi R^4 \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/6}$$

$$= \frac{\pi R^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

S_2



On S_2 θ is const $\theta = \theta_0 = \frac{\pi}{6}$

$$\Rightarrow |d\vec{A}| = ds_r(\theta_0)\, ds_\phi(\theta_0)$$

$$= dr\; r\sin\theta_0\, d\phi$$

$$= \frac{r}{2}\, dr\, d\phi$$

$$\Rightarrow \int_{S_2} \vec{v}\cdot d\vec{A} = \int \vec{v}\cdot\hat{\theta}\, dA$$

$$= \int_{S_2} v_\theta(r,\theta_0,\phi)\, \frac{r}{2}\, dr\, \phi$$

$$= \frac{1}{2}\int_0^{2\pi} d\phi \int_0^R dr\; 4r^3 \underbrace{\cos\theta_0}_{=\sqrt{3}/2}$$

$$= \sqrt{3}\,(2\pi)\left(\frac{R^4}{4}\right)$$

$$= \frac{\sqrt{3}}{2}\pi R^4$$

$$\therefore \oint_S \vec{v}\cdot d\vec{A} = \int_{S_1} \vec{v}\cdot d\vec{A} + \int_{S_2} \vec{v}\cdot d\vec{A} = \boxed{\frac{\pi R^4}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)}$$

Problem 3 Continued

Now consider volume integral of $\vec{\nabla}\cdot\vec{v}$

Using the expression for divergence in spherical coordinates given in the inside cover of "Griffiths"

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(4r^2\cos\theta\sin\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\underbrace{(r\tan\theta)}_{0}$$

$$= 4r\sin\theta + \frac{4r}{\sin\theta}(\cos^2\theta - \sin^2\theta) = 4r\frac{\cos^2\theta}{\sin\theta}$$

In spherical coordinates, the volume element is

$$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\therefore \int_V (\vec{\nabla}\cdot\vec{v})\, dV = \int_0^{2\pi} d\phi \int_0^{\pi/6} d\theta \int_0^R dr \left(4r\frac{\cos^2\theta}{\sin\theta}\right)(r^2\sin\theta)$$

$$= 4\int_0^{2\pi} d\phi \int_0^{\pi/6} d\theta\, \cos^2\theta \int_0^R dr\, r^3$$

$$= 4(2\pi)\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)\left(\frac{R^4}{4}\right)$$

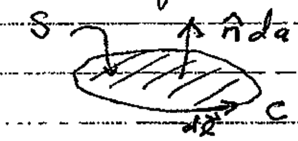
$$\int_V (\vec{\nabla}\cdot\vec{v})\cdot d\bar{V} = \boxed{\frac{\pi R^4}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)} = \oint_S \vec{v}\cdot d\vec{A}$$

q.e.d.

(4) Geometrical interpretation of
$\vec{\nabla}\times(\vec{\nabla}\phi) = 0$ and $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

(a) Consider the surface integral

$$\int_S \vec{\nabla}\times(\vec{\nabla}\phi)\cdot d\vec{a}$$



According to Stokes' theorem this equals the contour integral

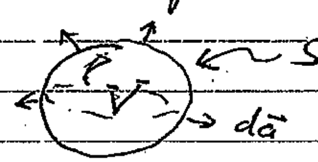
$$\oint_C \vec{\nabla}\phi\cdot d\vec{l}$$ where C is the boundary of S

But according to the fundamental theorem on line integrals, the is equal to difference of values of ϕ at the endpoints of the curve

$$\oint_C \vec{\nabla}\phi\cdot d\vec{l} = \phi(\vec{r}_2) - \phi(\vec{r}_1)$$. But $\vec{r}_1 = \vec{r}_2$ for a <u>closed curve</u>. The boundary is a points $\Rightarrow \int_S \vec{\nabla}\times(\vec{\nabla}\phi)\cdot d\vec{a} = 0$. Since this holds true for any open surface $\vec{\nabla}\times(\vec{\nabla}\phi) = 0 \quad \forall\phi$

(b) Consider the volume integral

$$\int_V \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\,d^3r$$



Div theorem $\Rightarrow \oint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a}$

(Next Page)

Now take the surface as a big "baloon" bounded by a small countour (the neck of the baloon)

By Stokes' Theorem

$$\oint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_c \vec{A}\cdot d\vec{l}$$

However, for the closed surface, the contour shrinks to a point

$$\Rightarrow \oint_{c\to 0} \vec{A}\cdot d\vec{l} = 0$$

Again this is true for an arbitrary volume V so $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ for any vector field $\vec{A}$

Both of these result from the geometry: "The boundary of a boundary is a point"