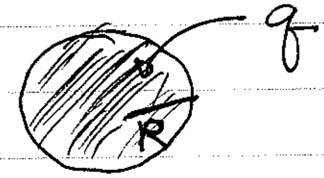
Physics 405 Problem Set #5

Solutions

(1) Consider a uniformly charged ball of radius R



(a) Work necessary to establish charge distribution

$$W = \frac{1}{2}\int_{\substack{\text{all}\\ \text{space}}} V(\vec{r})\,\rho(\vec{r})\,d^3r$$

Here $\rho(\vec{r}) = \begin{cases} \dfrac{q}{\frac{4}{3}\pi R^3} & r < R \\ 0 & r > R \end{cases}$

And because of spherical symmetry

$$V(\vec{r}) = V(r) \quad (\text{independent of } \theta, \phi)$$

$$\Rightarrow \quad W = \frac{1}{2}\left(\frac{q}{\frac{4}{3}\pi R^3}\right)\int_0^R \underbrace{4\pi r^2\,dr}_{= d^3r \text{ for spherical symmetry}} V(r)$$

Now we need $V(r)$.

Because of the spherical symmetry, the easiest route to finding $V(r)$ is to use Gauss's Law to first find $\vec{E}(\vec{r})$ and the integral $V(r) = -\int_{\infty}^{r} E(r)\,dr$

From Gauss's Law: (symmetry $\vec{E}(\vec{r}) = E(r)\hat{r}$

$$\oint_{\text{sphere of radius } R} \vec{E}\cdot d\vec{a} = 4\pi r^2 E(r) = \frac{Q_{enc}(r)}{\epsilon_0}$$

$$Q_{enc}(r) = \begin{cases} \left(\frac{q}{\frac{4}{3}\pi R^3}\right)\frac{4}{3}\pi r^3 = q\left(\frac{r}{R}\right)^3 & r \le R \\ q & r \ge R \end{cases}$$

$$\Rightarrow E(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\,\frac{q\,r}{R^3} & r \le R \\ \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2} & r \ge R \end{cases}$$

for $r \le R$

$$\therefore V(r) = \int_r^{\infty} E(r)\,dr = \int_r^{R} \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\,r\,dr + \int_R^{\infty}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R^3}\left(\frac{R^2}{2} - \frac{r^2}{2}\right) + \frac{1}{R}\right]$$

$$V(r) = \frac{q}{8\pi\epsilon_0}\left(\frac{-r^2 + 3R^2}{R^3}\right) \quad r < R$$

$$\therefore W = \frac{1}{2}\left(\frac{q}{\frac{4}{3}\pi R^3}\right)\left(\frac{q}{8\pi\epsilon_0}\right)\frac{4\pi}{R^3}\int_0^R dr\; r^2\left(3R^2 - r^2\right)$$

$$= \frac{q^2}{\frac{16\pi}{3} R^6 \epsilon_0}\left[R^2 r^3 - \frac{1}{5} r^5\right]_0^R$$

$$\Rightarrow \boxed{W = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right)}$$

Units are correct!

(b) Now use

$$W = \frac{\epsilon_0}{2}\int |\vec{E}(r)|^2\, d^3r$$

$$= \frac{\epsilon_0}{2}\int_0^\infty (E(r))^2\, 4\pi r^2\, dr$$

$$= \frac{\epsilon_0}{2}\left[\int_0^R \left(\frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\right)^2 4\pi r^2 dr + \int_R^\infty \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2 4\pi r^2 dr\right]$$

$$= \frac{1}{8\pi\epsilon_0}\left[\frac{q^2}{R^6}\int_0^R r^4\, dr + q^2\int_R^\infty \frac{dr}{r^2}\right]$$

$$= \frac{1}{8\pi\epsilon_0}\left[\frac{q^2}{5R} + \frac{q^3}{R}\right]$$

$$\Rightarrow W = \frac{1}{4\pi\epsilon_0}\left[\frac{3}{5}\frac{q^2}{R}\right]$$

(c) We now want to show that

$$\lim_{\text{surface} \to \infty} \oint_S \frac{\epsilon_0}{2} V(\vec{r})\, \vec{E}(\vec{r}) \cdot d\vec{a} \rightarrow 0$$

$$\lim_{S \to \infty} \Rightarrow \frac{\epsilon_0}{2} \oint_S V(r)\, \vec{E}(r) \cdot d\vec{a}$$

$$\lim_{a \to \infty} \Rightarrow \frac{\epsilon_0}{2} V(r=a)\; E(r=a)\; 4\pi a^2$$

Outside ball

$$V(a) = \frac{1}{4\pi\epsilon_0} \frac{q}{a} \qquad E(a) = \frac{1}{4\pi\epsilon_0} \frac{q}{a^2}$$

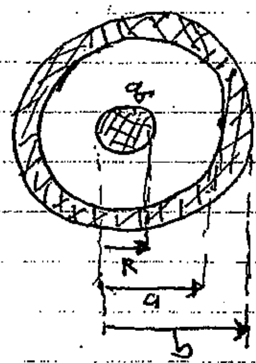
$$\Rightarrow \oint \frac{\epsilon_0}{2} V(\vec{r})\, \vec{E}(\vec{r}) \cdot d\vec{a} = \frac{1}{8\pi\epsilon_0} \frac{q^2}{a}$$

$$\boxed{\lim_{a \to \infty} \Longrightarrow 0}$$

So in e.q. (2.44) from Griffiths

$$W = \frac{\epsilon_0}{2} \int E^2\, d^3r \quad \text{as expected}$$

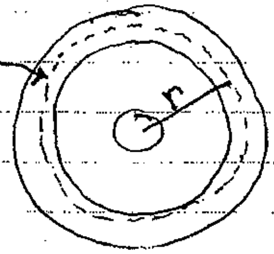
(2) Griffiths, Problem 2.35



Change q on the surface of a conducting sphere of radius R surrounded by a conducting spherical shell, inner radius a, outer radius b

(a) The field inside the conductor is zero. Therefore, by Gauss's Law:

Gaussian surface a radius r, $a<r<b$



$$\underbrace{4\pi r^2 E(r)}_{\substack{\text{Flux of electric field} \\ \text{(due to spherical sym.)}}} = \frac{1}{\epsilon_0} Q_{enc} = \frac{1}{\epsilon_0}\left(q + \underbrace{\sigma_a 4\pi a^2}_{\text{induced charge}}\right) = 0 \text{ (inside shell)}$$

$$\Rightarrow \boxed{\sigma_a = -\frac{q}{4\pi a^2}}$$

Because the conductor is overall neutral,

$$\underbrace{4\pi a^2 \sigma_a}_{\substack{\text{Total charge} \\ \text{at } r=a}} = \underbrace{-4\pi b^2 \sigma_b}_{\substack{-(\text{Total charge} \\ \text{a } r=b)}} \Longrightarrow \boxed{\sigma_b = \frac{q}{4\pi b^2}}$$

(b) $V(0) = -\int_{\infty}^{0} E(r)\,dr$ (spherical symmetry, $r=\infty$ as groun[d]

Using Gauss' Law, for spherical symmetry

$$E(r) = \frac{Q_{enc}(r)}{4\pi\epsilon_0 r^2}, \quad Q_{enc}(r) = \begin{cases} 0 & r<R \\ q & R<r<a \\ 0 & a<r<b \\ q & r>b \end{cases}$$

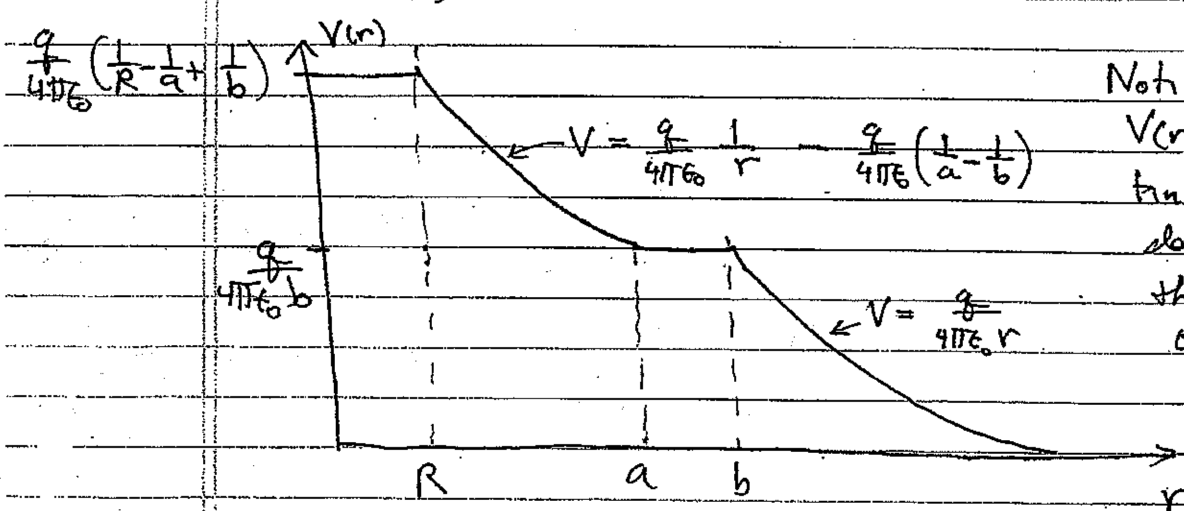
$$\Rightarrow E(r) = \begin{cases} 0 & r<R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} & R<r<a \\ 0 & a<r<b \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} & r>b \end{cases}$$

(Check that this satisfies the boundary conditions!)

$$\Rightarrow V(0) = \int_R^a \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr + \int_b^{\infty} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr$$

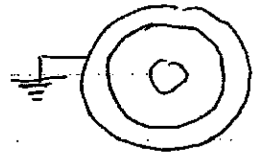
$$\boxed{V(0) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{a} + \frac{1}{b}\right)}$$

A general graph of the potential as a function of r is:



Notice that V(r) has discontinuity in its derivative at the surfaces of the conduc[tors] (Does this make sense

(c) The outer surface is grounded $\Rightarrow$ $V(b) = V(\infty) = 0$



What changes?

- We must maintain $\vec{E} = 0$ inside conductor
- The outer shell is no longer necessarily electrically neutral since charges can be exchanged with ground

$\sigma_a = -\frac{q}{4\pi a^2}$ (required to keep $E = 0$ inside shell)

No change

$\sigma_b = 0$ (required to keep $V = 0 = V(\infty)$)

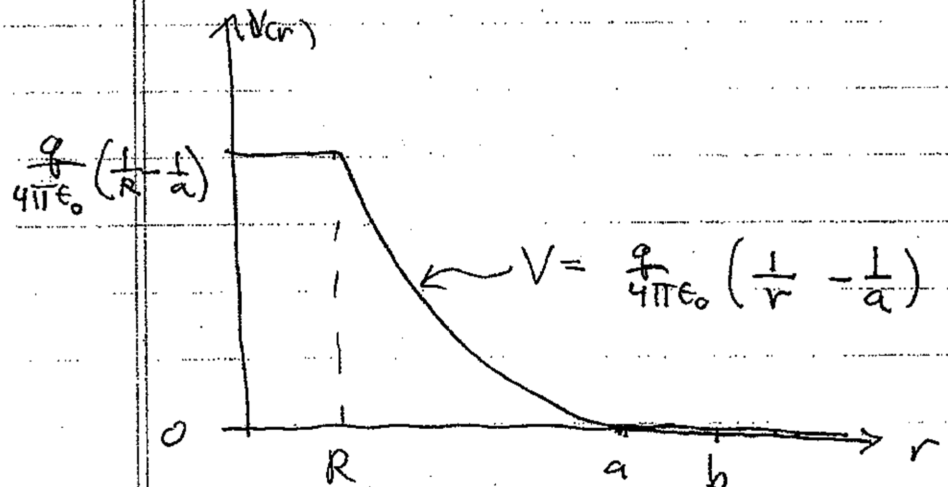
this changes!

Now $V(0) = -\int_\infty^0 E(r)\,dr = -\int_a^0 E(r)\,dr$ (since $V = 0$ for $r > a$)

$= -\int_a^R E(r)\,dr = -\frac{q}{4\pi\epsilon_0}\int_a^R \frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{a}\right)$

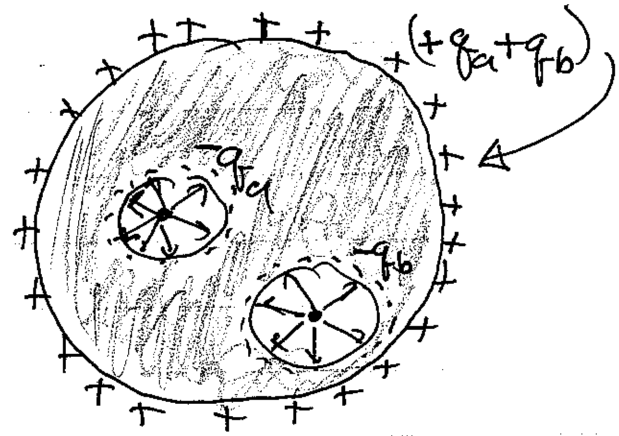
this changes!

New graph of potential: $V_{New} = \begin{cases} V_{old} - \frac{q}{4\pi\epsilon_0 b} & r \\ 0 & r \geq \end{cases}$



(3) Griffiths 2.36

(a)



Since $\vec{E}=0$ inside the bulk of the conductor, charges must be drawn to the surface of the cavity to shield the electric field of q_a and q_b (here assumed positive). The conductor is overall neutral, so the net charge $+q_a+q_b$ must then reside on the outer surface of sphere.

(a) $\boxed{\sigma_a = \frac{-q_a}{4\pi a^2}}$, $\boxed{\sigma_b = \frac{-q_b}{4\pi b^2}}$

$$\boxed{\sigma_R = \frac{+q_a+q_b}{4\pi R^2}}$$

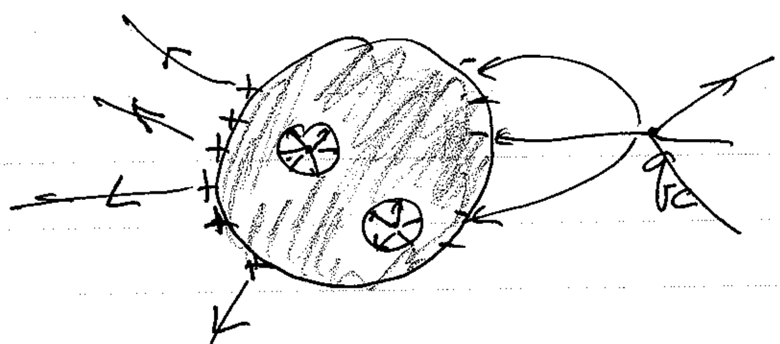
(b) Using Gauss's law. We know surface @ R is equipotential $\Rightarrow$ $+q_a+q_b$ uniform on shell $\Rightarrow$ $\boxed{\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{q_a+q_b}{r^2}\hat{r}}$ $r>R$

(c) Field within each cavity = electric field of a point charge

$$\boxed{\begin{array}{ll} \vec{E} = \dfrac{1}{4\pi\epsilon_0} \dfrac{q_a}{\mathscr{r}_a} \hat{\mathscr{r}}_a & \vec{\mathscr{r}}_a = \vec{r} - \vec{r}_a \\ \vec{E} = \dfrac{1}{4\pi\epsilon_0} \dfrac{q_b}{\mathscr{r}_b} \hat{\mathscr{r}}_b & \vec{\mathscr{r}}_b = \vec{r} - \vec{r}_b \end{array}}$$

(d) Since the surface charge on the cavity shields q_a from q_b, they exert no force on one another

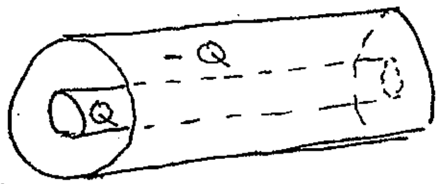
(e) If an external charge q_c is brought near to the conductor, the surface charges @ R must adjust themselves to ensure $\hat{E} = 0$ in the bulk.



Thus, the only thing that changes is charge density @ R (σ_R)

Everything else is unchange since charges are shielded

(4) Griffiths, Problem 2.39



Find capitence/length

To find the capitance we distribute charge $+Q$ on the surface of one conductor and charge $-Q$ on the surface of the other and find the potential difference: $V_{+-} = -\int_{(+)}^{(-)} \vec{E} \cdot d\vec{l}$

For a long cylinder $L >> b$, we can ignore fringing fields and approximate the cylinder as infinite.

By Gauss' Law:

$$\underbrace{(2\pi r L)\, E(r)}_{= \oint \vec{E} \cdot d\vec{A} \text{ (for cylindrical symmetry)}} = \frac{Q_{enc}(r)}{\epsilon_0} \Rightarrow E(r) = \frac{1}{2\pi\epsilon_0}\left(\frac{\lambda_{enc}(r)}{r}\right)$$

here $\lambda_{enc}(r) \equiv \frac{Q_{enc}}{L}$ = Charge per unit length enclose in the imagined Gaussian cylinder, radius r

For the problem at hand:

$$\lambda_{enc}(r) = \begin{cases} 0 & r < a \\ \frac{Q}{L} & a < r < b \\ 0 & r > b \end{cases}$$ (equal + opposite charges cancel.

$$\Rightarrow E(r) = \begin{cases} 0 & r < a \\ \frac{1}{L} \cdot \frac{1}{2\pi\epsilon_0} \frac{Q}{r} & a < r < b \\ 0 & r > b \end{cases}$$

The potential difference V_{+-} is then

$$V_{+-} = -\int_b^a E(r)\,dr = \frac{Q}{2\pi\epsilon_0 L}\left(\int_a^b \frac{dr}{r}\right)$$

$$= \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$$

By definition, $C = \frac{Q}{V_{+-}}$ (the capacitance)

$\therefore$ the capacitance/length

$$\boxed{C/L = \frac{Q/L}{V_{+-}} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}}$$

The total potential energy stored in the field can be calculated by

$$U = \frac{\epsilon_0}{2}\int_{\text{All space}} d^3r\,|\vec{E}(\vec{r})|^2, \quad \text{with } d^3r = 2\pi r\,dr\,L$$

(cylindrical coordinates)

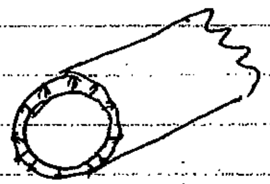
Since $\vec{E} = 0$ except for $a<r<b$

$$\Rightarrow U = \frac{\epsilon_0}{2}\left(\frac{Q}{2\pi\epsilon_0 L}\right)^2 2\pi L\int_a^b \frac{r\,dr}{r^2}$$

$$= \frac{Q^2}{8\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right) = \frac{1}{2}\underbrace{\left(\frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}\right)}_{C}\underbrace{\left(\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)\right)^2}_{V^2}$$

$= \frac{1}{2}CV^2$ = Work necessary to assemble charges on the conductors (Read Griffiths)

In the limit that the spacing between the cylinders goes to zero, we expect the field between to become more and more uniform, and so the coaxial should look like a parallel plate

Field almost uniform inside 

Let $b = a + d$

$$\frac{d}{a} \ll 1$$

$$\Rightarrow C/L = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} = \frac{2\pi\epsilon_0}{\ln\left(1+\frac{d}{a}\right)}$$

Now use the first order Taylor expansion

$$\ln(1+\delta) \approx \delta \quad \text{for } \delta \ll 1$$

$$\Rightarrow \frac{C}{L} \approx \frac{2\pi\epsilon_0}{d/a} \Rightarrow C \approx \epsilon_0 \frac{2\pi L a}{d}$$

Aside $2\pi a L$ = Area of overlap $\equiv A$

$$\Rightarrow C \approx \epsilon_0 \frac{A}{d} \quad \text{as expected}$$