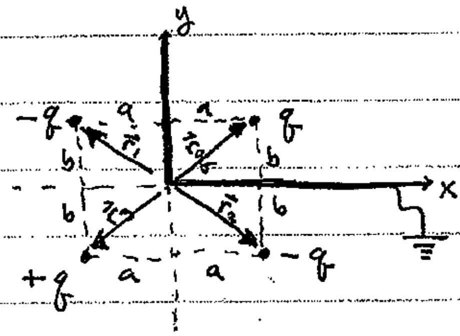
Physics 405. P.S.#6 Solutions

(1) Griffiths Problem 3.10



Finding the image charges is more of an art than a science. That is must "guess" the answer an then check it.

From the classic image problem of a point charge above a single plane we know that Laplace's Eq, and the boundary conditions are satisfied, so you might have chosen the 2 negative charges in the 2^{nd} and 3^{rd} quadrant. However, you will find that the potential is not zero on the planes. We need a 3^{rd} charge, $+q$, in the 3^{rd} quadrant

Configuration of image charges

$-q$ at $\vec{r}_1 = -a\hat{x} + b\hat{y}$ and $\vec{r}_2 = +a\hat{x} - b\hat{y}$

$+q$ at $\vec{r}_3 = -a\hat{x} - b\hat{y}$

The potential is that of the real charge plus 3 "images"

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r}-\vec{r}_q|} - \frac{q}{|\vec{r}-\vec{r}_1|} - \frac{q}{|\vec{r}-\vec{r}_2|} + \frac{q}{|\vec{r}-\vec{r}_3|}\right)$$

$$\Rightarrow \boxed{V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right)}$$

Check:

- $\nabla^2 V = 0$ since we constructed it from such solutions
- Boundary conditions

(1) There is a charge at $\vec{r}_q = a\hat{x} + b\hat{y}$ by construction

(2) $$V(x,0,z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+y^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+y^2+z^2}}\right) = 0 \checkmark$$

(3) $V(0,y,z) = 0 \checkmark$

The force on q is the electric field at q due to the surface charge on the conducting planes times q. But the E-field at $\vec{r}_q$ due to the surface charge is <u>exactly</u> mimicked by the three image charges. Thus we can use Coulomb's law: $\vec{F}(\vec{r}_q) = q\,\vec{E}(\vec{r}_q)$

$$\vec{F}(\vec{r}_q) = \frac{q}{4\pi\epsilon_0}\left\{\frac{-q\,(\vec{r}_q - \vec{r}_1)}{|\vec{r}_q - \vec{r}_1|^3} + \frac{-q\,(\vec{r}_q - \vec{r}_2)}{|\vec{r}_q - \vec{r}_2|^3} + \frac{q\,(\vec{r}_q - \vec{r}_3)}{|\vec{r}_q - \vec{r}_3|^3}\right)$$

Plugging in for $\vec{r}_q, \vec{r}_1, \vec{r}_2, \vec{r}_3$ we get

$$\boxed{\begin{aligned}\vec{F}(\vec{r}_q) &= \frac{q^2}{16\pi\epsilon_0}\left\{-\frac{\hat{x}}{a^2} - \frac{\hat{y}}{b^2} + \frac{a\hat{x} + b\hat{y}}{(a^2+b^2)^{3/2}}\right\} \\ &= \frac{-q^2}{16\pi\epsilon_0}\left\{\left(1 - \frac{1}{\left(1+\frac{b^2}{a^2}\right)^{3/2}}\right)\frac{\hat{x}}{a^2} + \left(1 - \frac{1}{\left(1+\frac{a^2}{b^2}\right)^{3/2}}\right)\frac{\hat{y}}{b^2}\right\}\end{aligned}}$$

The work necessary to bring q in from ∞ is just ~~the work needed~~ $\frac{1}{4}$ of the electrostatic energy of the 4 charge system since it takes no work to bring the image charges in from ∞ (see handout given in class)

$$\therefore\ W = \frac{1}{4}\left\{ \frac{1}{4\pi\epsilon_0}\frac{q q_1}{r_{q1}} + \frac{1}{4\pi\epsilon_0}\frac{q q_2}{r_{q2}} + \frac{1}{4\pi\epsilon_0}\frac{q q_3}{r_{q3}} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}} \right)$$

$$= \frac{1}{4}\left(\sum_{\substack{i,j \\ i \neq j}} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} \right)$$

Plugging in for the charges and the distances between them

$$W = \frac{q^2}{16\pi\epsilon_0}\left\{ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2+b^2}} + \frac{1}{2\sqrt{a^2+b^2}} - \frac{1}{2b} - \frac{1}{2a} \right.$$

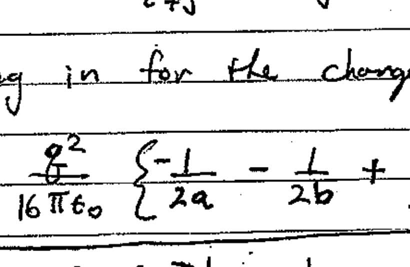
$$\Rightarrow \boxed{W = \frac{q^2}{16\pi\epsilon_0}\left\{ -\frac{1}{a} - \frac{1}{b} + \frac{1}{\sqrt{a^2+b^2}} \right\}}$$

(Check that $W = -\int \vec{F}\cdot d\vec{l}$ where $\vec{F}$ is the calculated for

The method will work if $180°$ is divisible by the angle between the planes.

i.e. $\Theta = \frac{180°}{n}$ $n = 1, 2, 3 \ldots$

Example $\theta = 60°$, $n = 3$

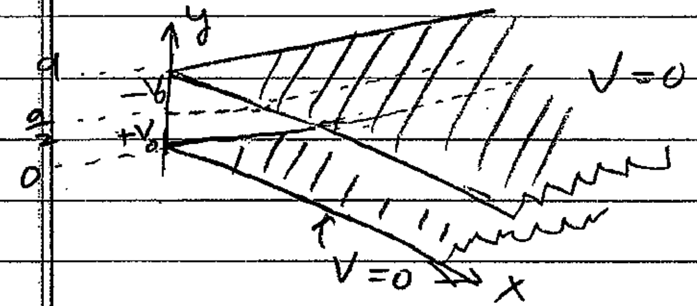


The solid circle is the true charge and the images are open circles. In general we need $n-1$ image charges

As long as the charges sit on the corners of a polygon ~~th~~ such that the conducting planes bisect the sides, the boundary conditions are satisfied

Problem 2: Griffiths 3.12

Infinite slit with a step potential along y



Same as example 3.1 in Griffiths,

except: $V_0(y) = \begin{cases} V_0 & 0<y<\frac{a}{2} \\ -V_0 & \frac{a}{2}<y<a \end{cases}$

With the boundary conditions $V=0$ at $y=0$, $y=a$, $x=\infty$, the general solution of Laplace's eqn $\nabla^2 V = 0$ is

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-k_n x} \sin(k_n y)$$

$$k_n = \frac{n\pi}{a}$$

To find the expansion coefficients, we must match @ $x=0$

$$V(0,y) = V_0(y) = \sum_{n=1}^{\infty} C_n \sin(k_n y)$$

"Project" onto the orthogonal function $\sin(k_m y)$

$$\Rightarrow \int_0^a V_0(y) \sin(k_m y)\, dy = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^a dy \sin(k_n y) \sin(k_m y)}_{\frac{a}{2}\delta_{nm}}$$

$$\Rightarrow C_m = \frac{2}{a}\int_0^a dy\, V_0(y) \sin(k_m y) =$$

$$= \frac{2V_0}{a}\left[\int_0^{a/2} dy \sin\left(\frac{m\pi y}{a}\right) - \int_{a/2}^{a} dy \sin\left(\frac{m\pi y}{a}\right)\right]$$

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$$\Rightarrow C_m = \frac{2}{\pi m} V_0 \left(-\cos\left(\frac{m\pi}{a} y\right)\Big|_0^{a/2} + \cos\left(\frac{m\pi}{a} y\right)\Big|_{a/2}^{a} \right)$$

$$= \frac{2}{\pi m} V_0 \left(1 + \cos(m\pi) - \cos\left(\frac{m\pi}{2}\right) \right)$$

$$\Longrightarrow \quad C_m = \begin{cases} \frac{8V_0}{\pi m} & \text{if} \quad m = 2, 6, 10, 14 \text{ etc} \\ 0 & \text{otherwise} \end{cases}$$

(Mathematica, to follow)

Define the function

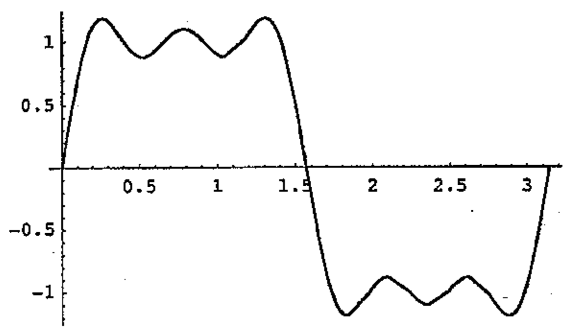
```
phi[x_,y_,m_] := Exp[-m x] Sin[m y]

c[m_] := 2/(m Pi)*(1+Cos[m Pi] - 2 Cos[m Pi/2])

V[x_,y_,Ntot_] := Sum[c[m]*phi[x,y,m], {m,1,Ntot}]
```

Plots of the potential (in units of V0) along the y-axis for as Ntot gets larger.

```
Plot[V[0,y,10],{y,0,Pi},PlotRange->All]
```

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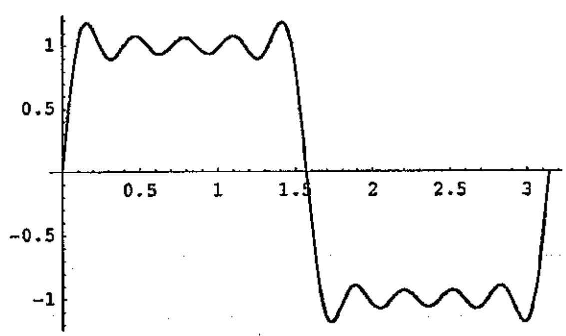
Plot[V[0,y,20],{y,0,Pi},PlotRange->All]
```

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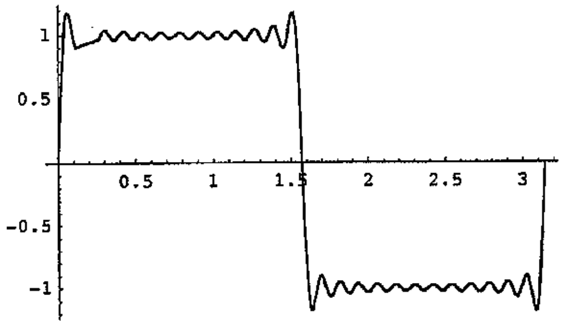
```
Plot[V[0,y,50],{y,0,Pi},PlotRange->All]
```



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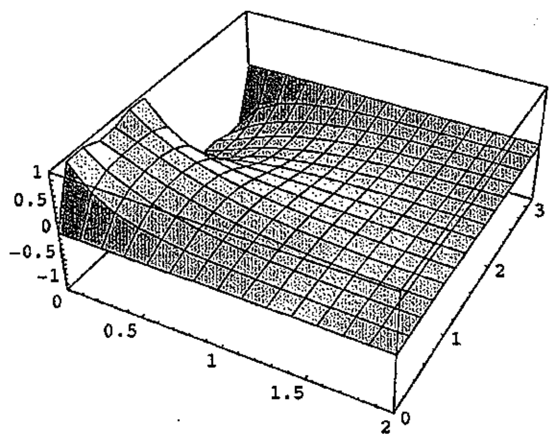
3D plot of the potential (in units of V0) for Ntot=20.

```
Plot3D[V[x,y,20],{x,0,2},{y,0,Pi},PlotRange->All]
```



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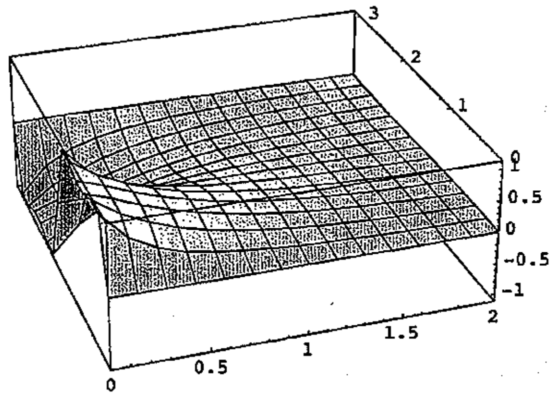
▪ different viewpoints

```
Show[%, ViewPoint->{-1.533, -4.596, 2.517}]
```

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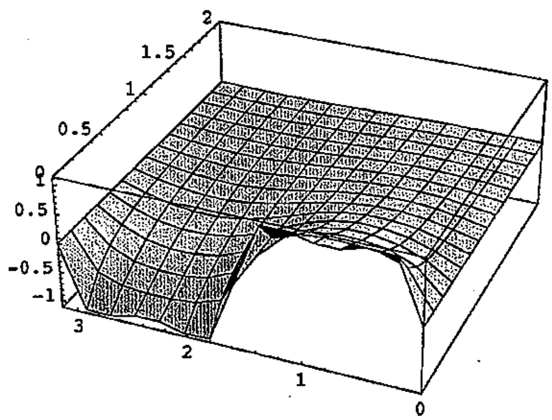
```
Show[%, ViewPoint->{-4.000, -1.670, 2.540}]
```

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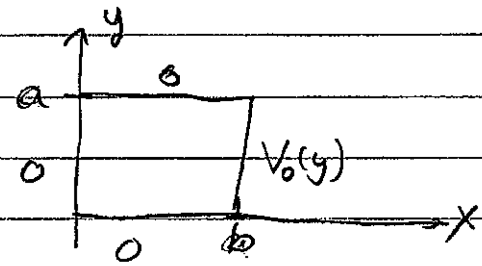
Problem: Griffiths 3.14

Rectangular pipe along z-axis with three sides grounded, and the fourth set @ $V_0(y)$



B.C.'s

(1) $V(0,y)=0$

(2) $V(x,0)=0$

(3) $V(x,a)=0$

(4) $V(y,b)=V_0(y)$

Separated solution to $\nabla^2 V=0$ in x and y

~~(I) $V_k(x,y)=(Ae^{+kx}+Be^{-kx})(C\sin ky+D\cos ky)$~~

(II) $V_k(x,y)=(Ae^{+ky}+Be^{-ky})(C\sin kx+D\cos kx)$

In order to satisfy B.C.'s (2) and (3), the potential must go to zero in two position along y

$\Rightarrow$ We must have an oscillating function of y

$\Rightarrow$ The "normal modes" must be of type (I)

$$V_k(x,y)=(Ae^{kx}+Be^{-kx})(C\sin ky+D\cos ky)$$

B.C. #1 $\Rightarrow V_k(0,y)=(A+B)(C\sin ky+D\cos ky)=0$

$$\Rightarrow A=-B$$

$$\Rightarrow V_k(x,y)=(C\sin ky+D\cos ky)\sinh(kx)$$

Where $\sinh(kx)=\dfrac{e^{kx}-e^{-kx}}{2}$, and I absorbed 2A into C + D

B.C.#2 $\quad V_k(x,0) = D\sin(kx) = 0$

$$\Rightarrow D = 0$$

B.C.#3 $\quad V_k(x,a) = C_k \sinh(kx)\sin(ka) = 0$

$$\Rightarrow \boxed{k_n = \frac{n\pi}{a}} \quad n = 1, 2, 3 \ldots$$

$\Rightarrow$ General solution, satisfying b.c.'s # 1–3

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} c_n \sinh(k_n x)\sin(k_n y)}$$

To find the coefficients "c_n", we use the final b.c.

B.C#4 $\quad V(b,y) = \sum_{n=1}^{\infty} C_n \sinh(k_n b) \sin(k_n y) = V_0(y)$

Use orthogonality of $\sin(k_n y)$: $\quad \int_0^a dy \sin(k_n y)\sin(k_m y) = \frac{a}{2}\delta_{mn}$

$$\Rightarrow \boxed{C_n = \frac{2}{a\sinh(k_n b)} \int_0^a dy \sin(k_n y) V_0(y)}$$

(b) For $V_0(y) = V_0$ constant

$$C_n = \frac{2V_0}{a\sinh(k_n b)} \int_0^a dy \sin(k_n y) \leftarrow \begin{cases} 0 & n \text{ even} \\ \frac{2a}{n\pi} & n \text{ odd} \end{cases}$$

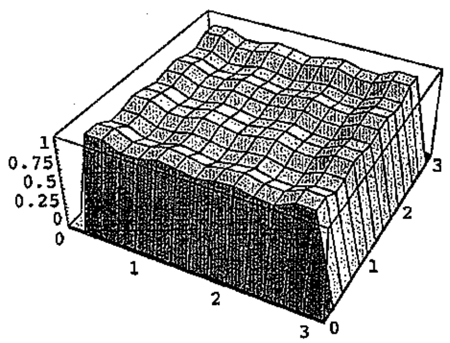
$$\Rightarrow \boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=1}^{\infty} \frac{\sinh\left(\frac{n\pi}{a}x\right)}{n\sinh\left(\frac{n\pi}{a}b\right)} \sin\left(\frac{n\pi}{a}y\right)}$$

Define the potential V(x,y,z)

```
V[x_,y_,z_,Ntot_] :=
     (16/Pi^2)* Sum[1/(n m) * Sin[n x]*Sin[m y]*
                    (Sinh[(n^2+m^2)^.5 z]/Sinh[(n^2+m^2)^.5 Pi]),
                    {n,1,Ntot,2},{m,1,Ntot,2}]
```

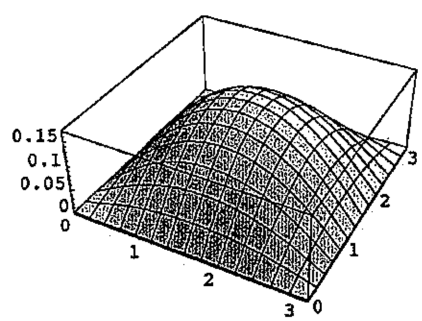
Plots of the potential in the z=π plane (should approach zero).

```
Plot3D[V[x,y,Pi,20],{x,0,Pi},{y,0,Pi},PlotRange->All]
```



Plot of the potential in the z=π /2 (x,y) plane.

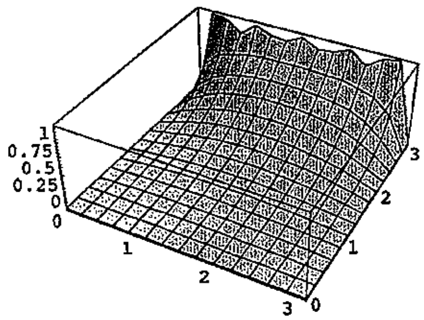
```
Plot3D[V[x,y,Pi/2,10],{x,0,Pi},{y,0,Pi},PlotRange->All]
```



**(* Note that the boundary conditions are satisfied in x and y.
This Plot seems to have a local maximum. How can this be? *)**

Plot of the potential in the y=π /2 (x,z) plane.

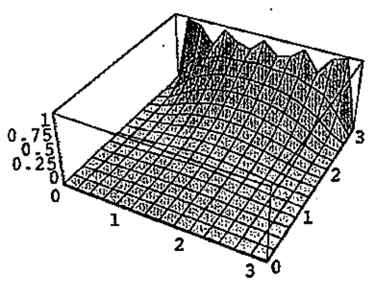
```
Plot3D[V[x,Pi/2,z,10],{x,0,Pi},{z,0,Pi},PlotRange->All]
```



(* Note that the boundary conditions are satisfied in x and z. *)

Plot of the potential in the (x = y,z) plane.

```
Plot3D[V[x,x,z,10],{x,0,Pi},{z,0,Pi},PlotRange->All]
```

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**(* This plot looks just like the previous one?
Why? This about x,y symmetry *)**