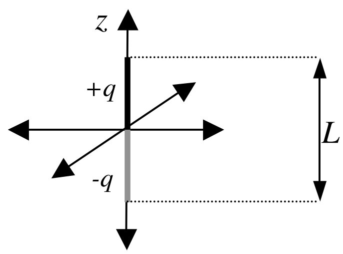
Physics 405

Problem Set #8: DUE Friday 4/3/2009

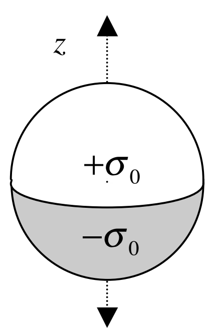
Problem 1.

(a) Consider an insulating rod of length L (negligible thickness). A charge q is uniformly distribution on the top half of the rod and charge $-q$ is uniformly distributed on the bottom half. The coordinate system is chosen below.



Find the monopole, dipole, and quadrupole moment of the distribution.

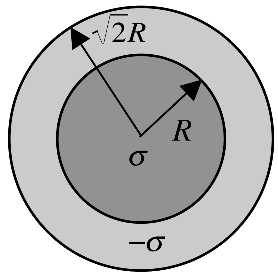
(b) Repeat this for a sphere, radius R, with surface charge density σ_0 on the northern hemisphere and $-\sigma_0$ on the southern hemisphere.



Compare you answer with the results from Griffith 3.22, in the previous problem set.

Problem 2.

Consider a disk of radius R and surface charge/area σ surrounded by an annulus of charge with outer radius $\sqrt{2}R$, inner radius R, and surface charge/area $-\sigma$ as sketched below.



(a) Find the multipole moments $Q^{(0)}, Q^{(1)}$, and $Q^{(2)}$ with the origin at the center of the disk and z-axis as the axis of symmetry. Give the asymptotic form of the potential, far from the disk, up to order $1/r^3$.

(b) Use direct integration to show that on the z-axis, the potential is

$$V(z) = \frac{\sigma}{2\varepsilon_0}\left(2\sqrt{z^2 + R^2} - \sqrt{z^2 + 2R^2} - z\right)$$

(c) Since the charge distribution is azimuthally symmetric, use this as a boundary condition at $\theta=0$, (i.e. $V(r,\theta = 0) = \frac{\sigma}{2\varepsilon_0}\left(2\sqrt{r^2 + R^2} - \sqrt{r^2 + 2R^2} - r\right)$) to find the solution to Laplace's equation in spherical coordinates for $r > \sqrt{2}R$ up to order $(R/r)^3$. With this solution, check your answer to (a).

Problem 3

In Lect. 17, by expanding the exact electrostatic potential $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$ without any assumption of symmetry, we found an expansion up to the dipole term,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_{tot}}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} + \ldots\right).$$

(a) Take this expansion to the next order, keeping all terms of order $(r'/r)^2$ in the integrand, and show that the quadrupole term can be written,

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\sum_{ij}\hat{r}_i\hat{r}_j\, Q_{ij}}{r^3},$$

where

$$Q_{ij} = \int \frac{1}{2}\left(3r_i'r_j' - (r')^2\delta_{ij}\right)\rho(\mathbf{r}')\, d^3r',$$

is the quadrupole tensor (matrix).

(b) Show that is matrix is symmetric (i.e. $Q_{ij} = Q_{ji}$) and traceless (i.e., its diagonal elements sum to zero).

(c) Show that for a charge distribution with azimuthal symmetry (with the z-axis the symmetry), the off diagonal elements vanish and $Q_{zz} = -2Q_{xx} = -2Q_{yy} = Q^{(2)}$, where $Q^{(2)}$ is the quadrupole moment we defined for azimuthal symmetry.

(d) Given the results to (c), show that the general quadrupole potential defined in (a) takes the form,

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q^{(2)}}{r^3}P_2(\cos\theta)$$

where r and θ are the spherical coordinates of the point of observation.