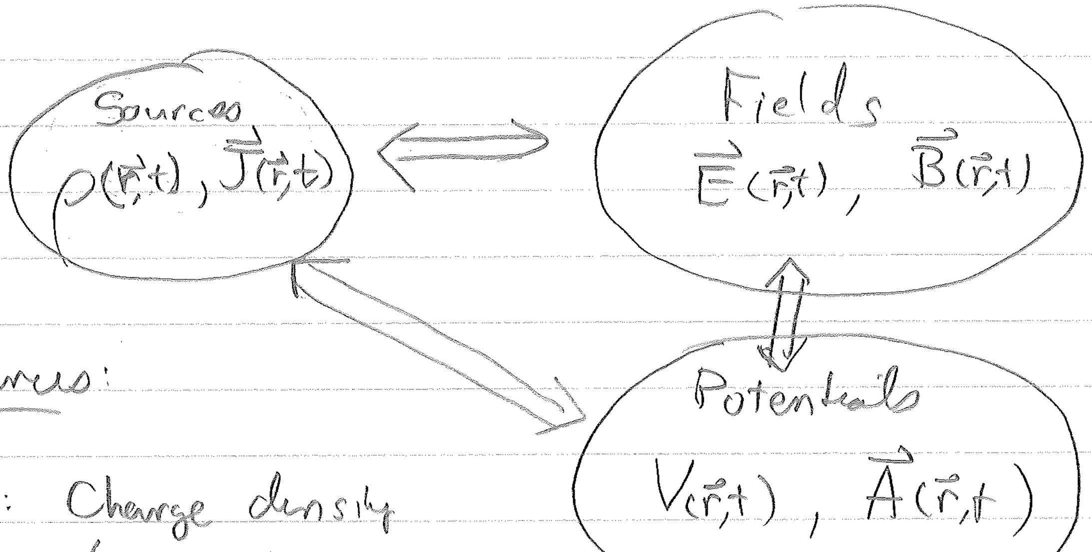
Physics 406: Electricity & Magnetism II

Department of Physics and Astronomy
University of New Mexico

Instructor: Prof. Ivan H. Deutsch

Lecture 1: Review of Electro/magneto Statics

Electricity and magnetism is the study of the fields generated by charges and currents and the forces on the matter due to the fields

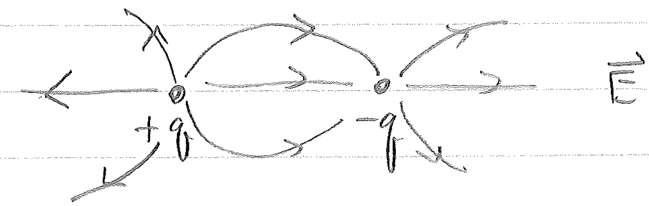


Sources:

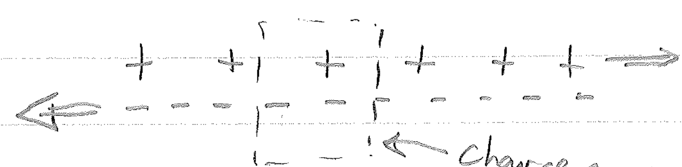
- $\rho(\vec{r},t)$: Charge density (Charge/Volume)
- $\vec{J}(\vec{r},t)$: Current density $\left(\frac{\text{Charge/time}}{\text{Area}}\right)$

Electrostatics: $\rho(\vec{r})$: independent of time

- Charges fixed in space



- Charges moving but in steady flow

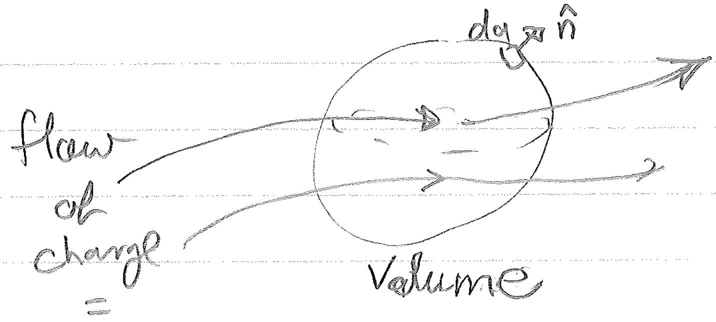


$$\frac{\partial \rho}{\partial t} = 0 \quad , \text{ but charges moving}$$

Magnetostatics $\vec{J}(\vec{r})$ independent of time

and $\frac{\partial \rho}{\partial t} = 0$ by $\frac{\partial \vec{E}}{\partial t} = 0$

$\Rightarrow$ Steady current:



Conservation of charge:

$$\underbrace{\oint \vec{J} \cdot \hat{n}\, da}_{\text{current out of surface}} = \underbrace{\frac{-dQ}{dt}}_{\text{change of charge in enclosed volume}}$$

Continuity Eq:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \left(= 0 \text{ magnetostatics} \right)$$

Field Equations (Sources generate fields)

Every vector field is defined by its divergence and curl everywhere in space @ all times

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \iff \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \quad \text{(Gauss)}$$

$$\vec{\nabla}\times\vec{E} = 0 \iff \oint_C \vec{E}\cdot d\vec{l} = 0 \quad \text{(conservative force)}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \iff \oint_S \vec{B}\cdot d\vec{a} = 0 \quad \text{(No magnetic monopoles)}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \iff \oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \quad \text{(Ampère)}$$

these together with the Lorentz force law

$$\vec{F} = \sum_i q_i \vec{E}(\vec{r}_i) + q_i \vec{v}_i \times \vec{B}(\vec{r}_i)$$

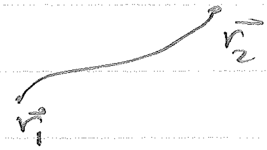
$$= \int \underbrace{d^3r}_{\text{Volume element}} \rho(\vec{r})\,\vec{E}(\vec{r}) + \vec{J}(\vec{r})\times\vec{B}(\vec{r})$$

define electromagneto statics

Potential formulation of Electrostatics

Electrostatic force is "conservative"

Work done on charge by field



$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = q \int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{r}$$

Around closed loop in electrostatics $\oint \vec{F}\cdot d\vec{l} = q \oint \vec{E}\cdot d\vec{l} = 0$

$$\Rightarrow \quad \vec{F} = -\vec{\nabla} U \,, \quad U(\vec{r}) = -\int_{\vec{r}_{ground}}^{\vec{r}} \vec{F}\cdot d\vec{l} = -q \int_{\vec{r}_g}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

U: potential energy of charge in field

Electrostatic potential $V(\vec{r}) = -\int_{\vec{r}_g}^{\vec{r}} \vec{E}\cdot d\vec{l}$

$$\boxed{\vec{\nabla}\times\vec{E} = 0 \iff \vec{E}(\vec{r}) = -\vec{\nabla} V}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\vec{\nabla} V = \boxed{-\nabla^2 V(\vec{r}) = \rho/\epsilon_0}$$

Poisson's eqn

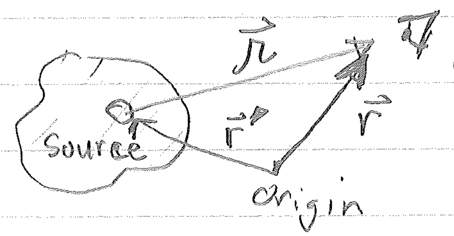
In a charge free region $\rho = 0$

$\nabla^2 V = 0$: Laplace's eqn:

Solution methods for electrostatics

- Symmetry: Gauss' Law $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$
- Coulomb's Law

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} \Rightarrow \vec{E} = \int d^3r'\,\frac{\rho(\vec{r}\,')\,\hat{\mathscr{r}}}{4\pi\epsilon_0\,\mathscr{r}^2}$$

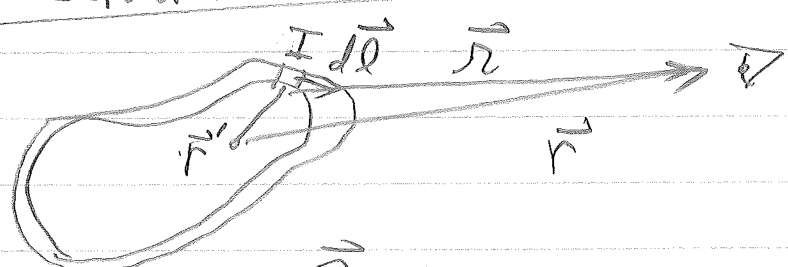


$$\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}\,'$$

- Boundary values / special functions (Laplace)

Magnetostatics

- Symmetry: Ampère's Law $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$
- Biot-Savart Law



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int I\,d\vec{l} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

-

Potential formulation of magnetostatics

$$\boxed{\vec{\nabla}\cdot\vec{B}=0 \iff \vec{B}=\vec{\nabla}\times\vec{A}}$$

$\vec{A}$ is the vector potential

- Gauge invariance: $\vec{A} \Rightarrow \vec{A}+\vec{\nabla}\chi \equiv \vec{A}'$

  $\vec{B} \Rightarrow \vec{B}$

$\Rightarrow$ $\vec{A}$ and $\vec{A}'$ yield the same physical $\vec{B}$

$\Rightarrow$ $\vec{\nabla}\cdot\vec{A}$ does not effect physical predictions

Gauge choice typical in magnetostatics $\vec{\nabla}\cdot\vec{A}=0$

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{A}}}_{0}) - \nabla^2\vec{A}$$ (important identity)

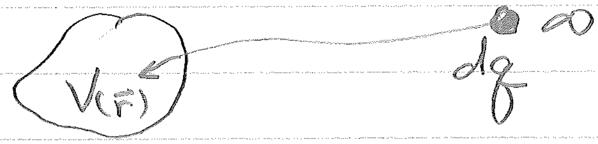
$\Rightarrow \nabla^2\vec{A} = -\mu_0\vec{J}(\vec{r})$ (Poisson equation for each Cartesian component)

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int d^3r' \,\vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$ (Biot-Savart)

Energy in fields = Work necessary to establish a charge distribution ρ

Electrostatics



$$dW = \underset{\substack{\uparrow \\ \text{no self energy}}}{\tfrac{1}{2}}\, dq\, V_{self} = \tfrac{1}{2}\rho(\vec{r})\, V(\vec{r})\, d^3r$$

$$\boxed{U_{Field} = \frac{1}{2}\int d^3r\ \rho(\vec{r})\, V(\vec{r})\, d^3r = \frac{\epsilon_0}{2}\int |\vec{E}|^2 d^3r}$$

Magnetostatics:

$\vec{B}$-fields do no work $\quad q\oint(\vec{v}\times\vec{B})\cdot d\vec{l} = 0$

$\vec{B}$ can change the direction of $\vec{v}$, but not its magnitude

Energy stored in a $\vec{B}$-field $\equiv$ Work necessary to establish a current distribution

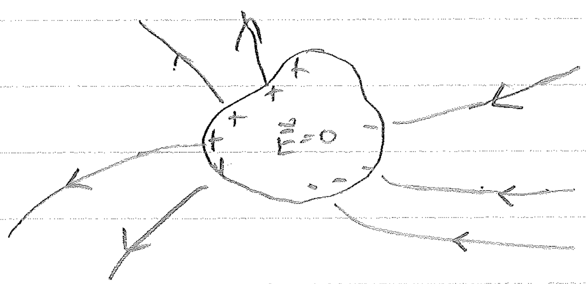
$\Rightarrow \vec{J}$ must change with time

Beyond magnetostatics (stay tuned)

Macroscopic Field Equations

Materials: Conductors / Insulators (Dielectrics)

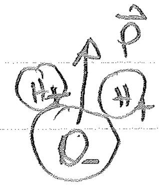
- Perfect conductors: Infinite mobility of free charges



In electrostatics, all free charges reside at the surface ($\vec{E}=0$ inside)

- Dielectrics: No free charge — Charges bound in neutral atoms / molecules / solids

H_2O



electric dipole $\vec{p} = q_+ \vec{r}_+ - q_- \vec{r}_-$

Polarization density $\vec{P} = \dfrac{\text{electric dipole}}{\text{volume}}$

- Bound charge density $\rho = -\vec{\nabla} \cdot \vec{P}$
- Displacement field $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$$\vec{\nabla} \cdot \vec{D} = \rho_{free}$$

susceptibility

- Linear response $\vec{P} = \epsilon_0 \chi \vec{E}$

$\vec{D} = \epsilon \vec{E}$, $\epsilon = \epsilon_0 (1 + \chi) =$ Dielectric permittivity

- <u>Magnetized material</u>: Induced magnetic dipoles in macroscopic material

$\vec{M} \equiv$ magnetic dipole / volume
(magnetization)

Three distinct classes arising from different microscopic origins (orbital motion of electrons, spin angular momentum)

- Diamagnetic $\vec{M}$ induced along $-\vec{B}$
- Paramagnetic $\vec{M}$ induced along $\vec{B}$
- Ferromagnetic macroscopic $\vec{M}$ w/o $\vec{B}$

Magnetization (bound) Current $\vec{J}_M = \vec{\nabla} \times \vec{M}$

Total current $\vec{J} = \vec{J}_{free} + \vec{J}_M$

$$\Rightarrow \vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J}_{free}$$

$$\Rightarrow \vec{\nabla} \times (\underbrace{\frac{1}{\mu_0}\vec{B} - \vec{M}}_{\vec{H}}) = \vec{J}_{free}$$

$\vec{H}$ "auxilory" magnetic field

$$\boxed{\vec{B} = \mu_0 \vec{H} + \vec{M}}$$