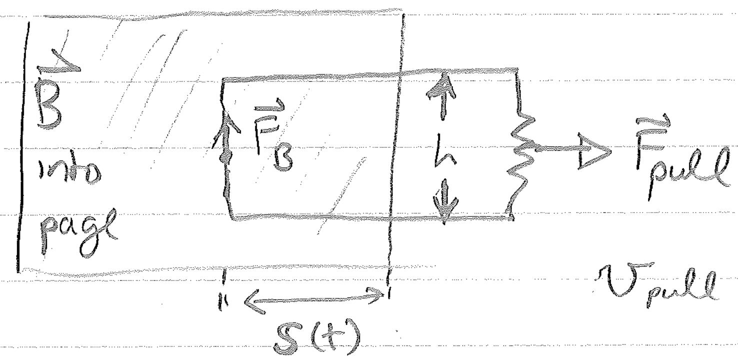
Physics 406: Lecture 3

Faraday's Law of Universal Induction

In the last lecture we saw that an EMF can be generated by moving a conductor through a magnetic field with speed v



$$v_{pull} = \frac{-ds}{dt}$$

$$\mathcal{E} = \oint \frac{\vec{F}_B \cdot d\vec{l}}{q} = vBh = \frac{-ds}{dt} h\, B$$

Note: Flux of magnetic field into the page

$$\Phi_B = \int \vec{B} \cdot \hat{n}_{\substack{\text{into}\\\text{page}}}\, da = B\, s(t)\, h$$

$$\Rightarrow \boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}}$$

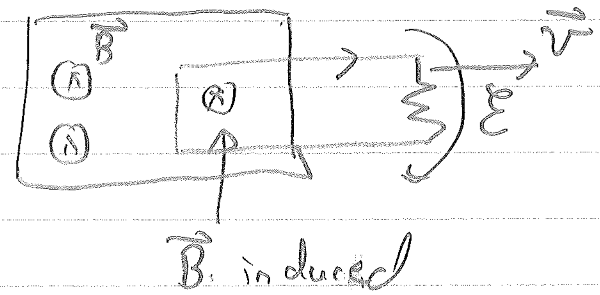
The EMF is related to the change in flux of $\vec{B}$ through the loop. The sign is determined by the right-hand-rule.

The negative sign in the law of induction is know as Lenz's law

$$\mathcal{E} = -\frac{d}{dt}\Phi_B$$

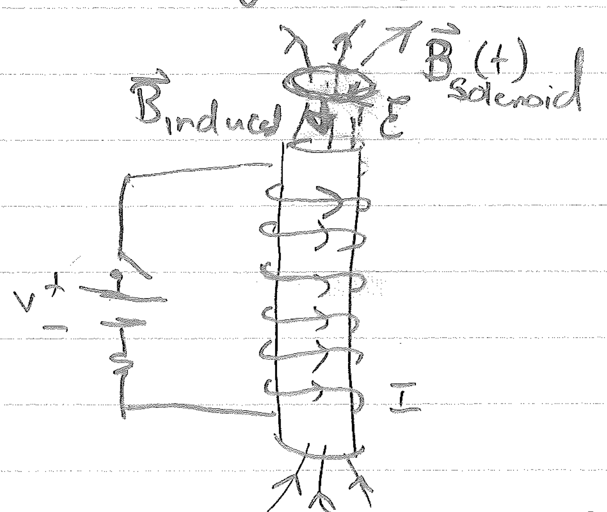
EMF establishes in such a way to keep the flux of $\vec{B}$ from changing

"Nature abhors a change in flux"



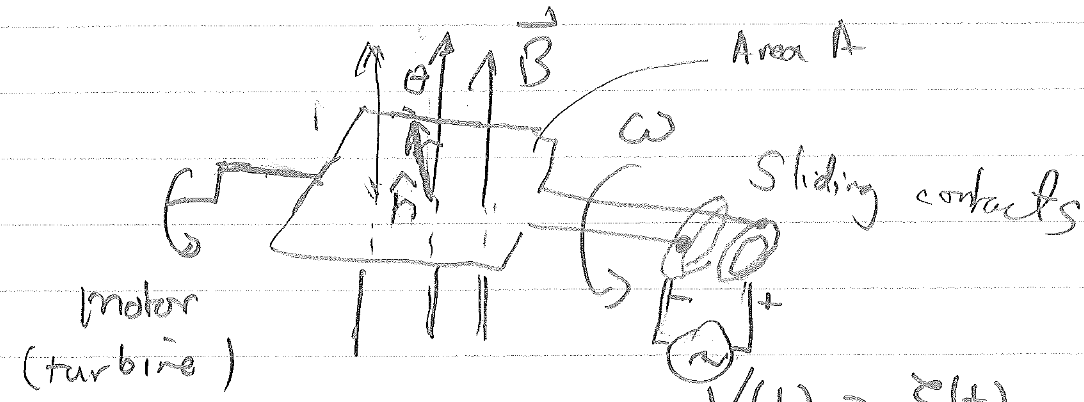
The EMF generates a magnetic field into the page because flux into page is decreasing (EMF replenishes it

Jumping ring trick:



When the switched is pulled, a magnetic flux shoot upward through ring. An $\mathcal{E}$ is induced to counter this which creates an opposite polarity $\vec{B}$ field in the ring which repells it from the solenoid

Example of Motional EMF: AC Generator



$V(t) = \mathcal{E}(t)$ Voltage

$$\Phi_B(t) = \int_S \vec{B}\cdot\hat{n}\,da = BA\cos\theta(t) = BA\cos\omega t$$

$$\boxed{V(t) = -\frac{d\Phi_B}{dt} = BA\omega\sin\omega t}$$ AC voltage

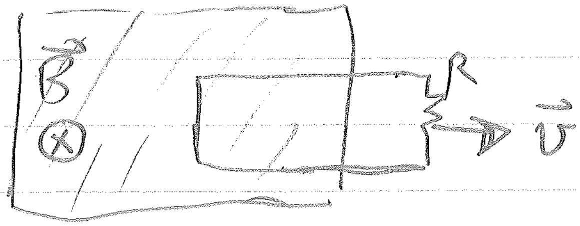
Numbers: $\nu = \frac{\omega}{2\pi} = 60$ cycles/sec $= 60$ Hz

Loop area $= (10\text{ cm})^2 = 1\text{ m}^2$

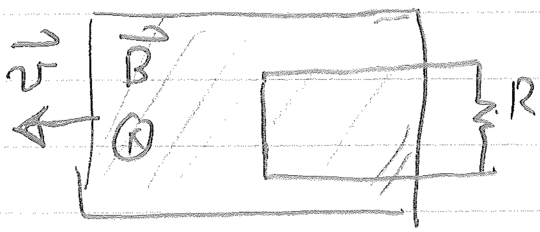
$B = 1$ Tesla

$$\Rightarrow \boxed{V_{max} = (2\pi)(60\text{ s}^{-1})(1\text{ T})(1\text{ m}^2) \cong 360 \text{ Volts}}$$

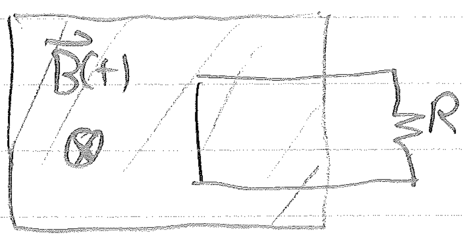
Faradays Universal Law



Case (i):
loop moved through $\vec{B}$ field region



Case (ii)
Region of $\vec{B}$ field moves w/ loop fixed



Case (iii)
Both the loop & Region of $\vec{B}$ fixed in space, but $\vec{B}$ changes w/ time

In all case

$$\boxed{\mathcal{E} = \oint d\vec{l} \cdot \frac{\vec{F}}{q} = -\frac{d}{dt} \oint \vec{B} \cdot \hat{n}\, da}$$

right hand rul

Of course Case (i) + (ii) are obvious from relativity, but this was not the thinking @ Faraday's time. Case (iii) is a real surprise.

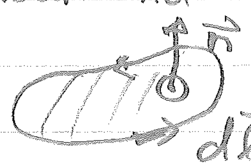
Einstein actually used Faraday's experiment to argue about relativity!

Induced Electric Field

In the rest frame of the loop (Cases (ii) & (iii) in the sketch of Faraday's experiment) the charges are at rest ⇒ No Lorentz force $q\vec{v}\times\vec{B}$. Therefore, the force on the charge must be due to an <u>electric field</u>

$$\mathcal{E} = \oint_{C} \vec{E}\cdot d\vec{l} = -\oint_{S} \frac{\partial \vec{B}}{\partial t}\cdot \hat{n}\, da$$

Contour and surface in fixed inertial frame



Right-hand rule

But, by Stokes's theorem $\oint_{C} \vec{E}\cdot d\vec{l} = \int_{S} (\vec{\nabla}\times\vec{E})\cdot\hat{n}\, da$

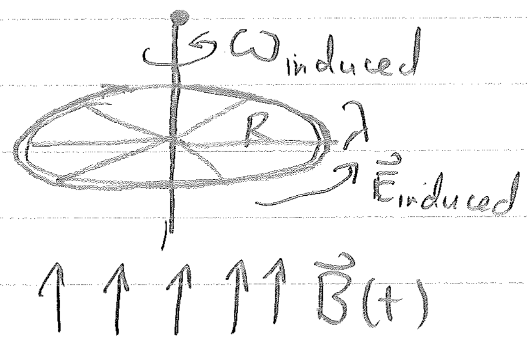
$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}}$$

Differential form of Faraday's Law

A changing $\vec{B}$-field at any point in space gives rise to a curling $\vec{E}$-field

⇒ Beyond electrostatics!

Example of Induced E-field



- A line of charge is glued to the rim of a wheel, radius R
- A uniform $\vec{B}$ field fills the space and then is reduced to 0

The wheel will rotate due to an induced $\vec{E}$-field:

$$\oint \vec{E}\cdot d\vec{l} = -\oint \vec{B}\cdot d\vec{a} = -\pi R^2 \frac{dB}{dt}$$

By symmetry, $|\vec{E}|$ is constant at given R and must by in the $\hat{\phi}$-direction

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = E(R)\, 2\pi R$$

$$\Rightarrow \boxed{\vec{E}(R) = -\frac{R}{2}\frac{dB}{dt}\hat{\phi}}$$

The force on an element of change λdl

$$d\vec{F} = \lambda dl\, \vec{E}$$

$\Rightarrow$ Torque on wheel $|\vec{\tau}| = |\vec{R}\times\vec{F}|$

$$\Rightarrow |\vec{\tau}| = \oint(\lambda\, dl)\, RE = 2\pi R\lambda\, RE$$

$$\Rightarrow \tau = \pi R^3 \lambda \frac{dB}{dt}$$

The instanteneous angular momentum of wheel

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow |\vec{L}| = \int_0^{t_f} |\vec{\tau}|\, dt$$

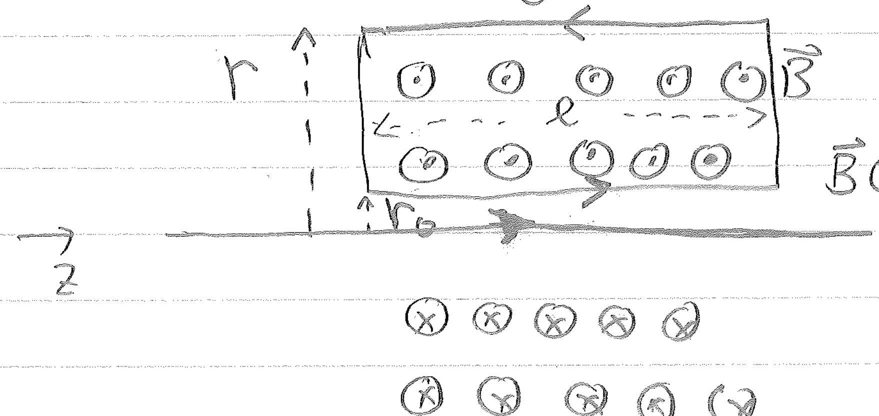
$$\Rightarrow |\vec{L}| = \int_{B_{initial}}^{B_{final}} |\vec{\tau}(B(t))| \frac{dt}{dB}\, dB = \int_{B_0}^{0} \pi R^3 \lambda \, dB$$

$$\boxed{|\vec{L}| = \lambda \pi R^3 B_0}$$

Where does the angular momentum come from?

Answer: The Electromagnetic field carries angular momentum (stay tuned)

Example: E-field induced around wires w/ changing currents



$$\vec{B}(r,t) = \frac{\mu_0 I(t)}{2\pi r}\hat{\phi}$$

(From Ampère's Law)

By symmetry, in analogy to what we know about a solenoid $\vec{E} = E(r)\hat{z}$

Choosing the contour as drawn

$$\oint_C \vec{E}\cdot d\vec{l} = E(r) - E(r_0) = -\frac{d}{dt}\Phi_B\Big|_{\text{enclosed}}$$

$$= -\frac{d}{dt}\int B(r)\,dr\,l = -l\frac{dI}{dt}\int_{r_0}^{r}\frac{dr}{r}\frac{\mu_0}{2\pi}$$

$$\Rightarrow \boxed{\vec{E}(r,t) = -\frac{\mu_0}{2\pi}\ln\left(\frac{r}{r_0}\right)\frac{dI}{dt}\hat{z}}$$

<u>Note:</u> $|\vec{E}| \to \infty$ as $r \to \infty$: <u>huh??</u>

This is unphysical

<u>Reason:</u> We derived $\vec{B}$ using Ampère's Law $\oint \vec{B}\cdot\hat{n}\,da = \mu_0\int \vec{J}\cdot\hat{n}\,da$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

But this is only exact when $\vec{\nabla}\cdot\vec{J} = 0$

$\Rightarrow$ <u>Steady current</u>

When $\frac{dI}{dt} \neq 0$ Ampère's Law must be modied. Our expression in

<u>quasi static</u> : Requires all distance

$r \ll c\,\tau_R$ Characteristic time scale