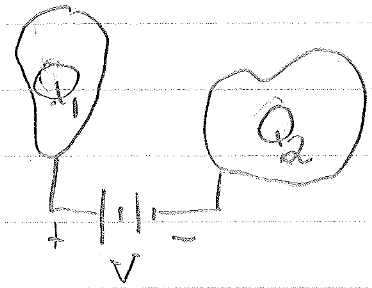
Physics 406: Lecture 4

Inductance and Energy in $\vec{B}$-field

Review

Recall, in electrostatics, the charge induced on one conductor due to the presence of another conductor is proportional to the potential difference between them



$$Q_1 = C_{12} V$$

Mutual capacitance

$$Q_2 = C_{21}(-V)$$

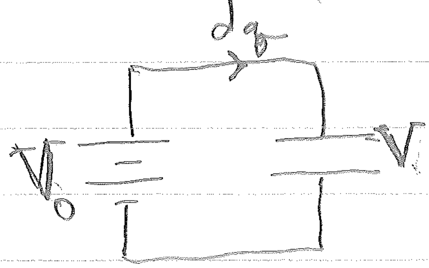
"Reciprocity" $C_{12} = C_{21} = C$

Self-Capacitance = Capacitance w.r.t. ground @ ∞

In quasistatics, changing $V \Rightarrow$ Changing Q

$$\frac{dQ}{dt} = I = C\frac{dV}{dt}$$

Work and Energy in Capacitor



Charging process

Work to add dq to Capacitor when potential is V

$$dW = dq\,V = C\,dV\,V$$

$\Rightarrow$ Total work

$$W = \int_0^{V_0} dW = \frac{1}{2}CV^2$$

The Work we do is stored as potential energy. We can take this potential energy as residing in the field.

Example: For a parallel plate capacitor

$$C = \frac{A \leftarrow \text{Area}}{\epsilon_0 d \leftarrow \text{distance}} \quad , \quad V = |\vec{E}|/d$$

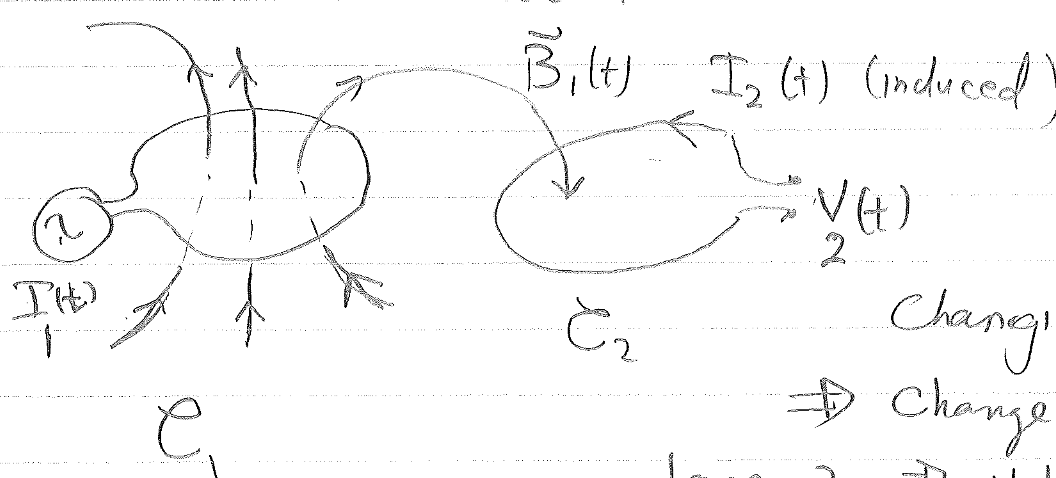
$$\Rightarrow U = \frac{1}{2} C V^2 = \frac{\epsilon_0}{2} |\vec{E}|^2 (\underbrace{A d}_{\text{volume}})$$

$\Rightarrow$ Energy density in field $u_E = \frac{\epsilon_0}{2} |\vec{E}|^2$

This is a general result for any $\vec{E}$-field

Inductance

The analogous phenomenon in magnetic field is inductance $\Rightarrow$ a changing current induces a voltage (for a capacitor, a changing voltage induces a current).



Changing $I_1(t)$
$\Rightarrow$ Change flux in loop-2 $\Rightarrow$ Voltage on C_2

The magnetic flux through contour-2 due to the instantaneous current flowing in loop-1

$$\Phi_{2,1} = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \oint_{C_2} \underbrace{\vec{A}_1}_{\text{vector potential}} \cdot d\vec{l}_2$$

$$\vec{B}_1 = \vec{\nabla} \times \vec{A}_1$$

But, according to the Biot-Savart Law

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint d\vec{l}_1 \times \frac{\hat{r}}{r^2} \quad \text{or} \quad \vec{A}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1}{r}$$

$$\Rightarrow \Phi_{2,1} = \underbrace{\left(\frac{\mu_0}{4\pi} \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r} \right)}_{M_{21} \,\equiv\, \text{Mutual Inductance}} I_1$$

(units $\equiv$ Henries)

Note reciprocity $\quad M_{1,2} = M_{21}$

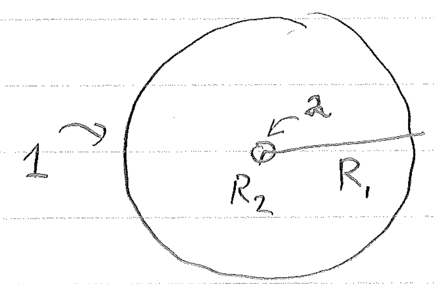
$\Phi_{2,1} = M_{21} I_1 \Rightarrow$ EMF on loop 2 due to changing $\vec{B}_1$

$$\boxed{V_2 = -\frac{d\Phi_{2,1}}{dt} = -M_{21} \frac{dI_1}{dt}}$$

Typically, to calculate the mutual inductance, we don't use the general formula. Instead, we assume a current I_1 and then calculate the flux Φ_{21} and then look at

$$\Phi_{21} = M_{21} I_1$$

Example: Two concentric, coplanar circular loops with radii $R_2 << R_1$



Find the Mutual inductance

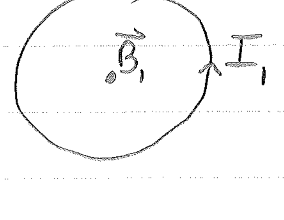
Using reciprocity, we will find the magnetic flux through loop 2 when a current flows through 1.

Because $R_2 << R_1$, we can approximate B_1 as uniform across loop 2

$$\Rightarrow \Phi_{2,1} \cong \left(\pi R_2^2\right) B_1(r=0)$$

Magnetic field @ origin due to a current in loop 1

Biot-Savart



$$\vec{B}_1 = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec{B}_1^{(0)} = \frac{\mu_0}{4\pi} I/R_1^2 \oint dl \; \hat{z}$$

$$\Rightarrow \vec{B}_1(0) = \frac{\mu_0}{4\pi}\frac{I}{R_1^2}(2\pi R_1)\hat{z} = \frac{\mu_0 I_1}{2R_1}\hat{z}$$

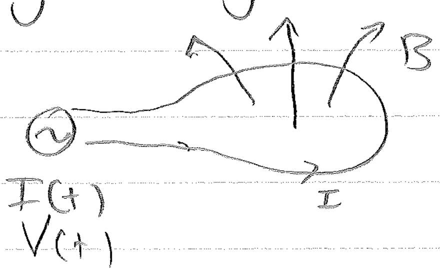
thus $$\Phi_{21} \cong \left(\frac{\mu_0 \pi R_2^2}{2R_1}\right) I_1$$

$$\Rightarrow \boxed{\text{Mutual inductance} \quad M_{2,1} = M_{1,2} \cong \frac{\mu_0 \pi R_2}{2R_1}}$$

Note: $[\mu_0]$ = Henry/meter $\qquad [\epsilon_0]$ = Farad/meter

Self inductance

A changing current in a loop not only affects an EMF around nearby loops, but a "back-EMF" on the loop itself to oppose the changing flux (Lenz's Law). The strength of the back-EMF is given by the self-inductance

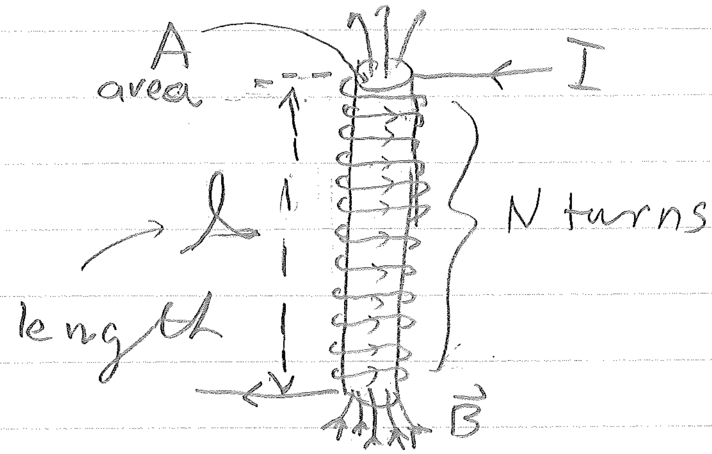


$$\Phi = L I$$

L ← self-inductance

Faradays law $\Rightarrow \boxed{V(t) = -L\frac{dI}{dt}}$

Example: Self-inductance of a solenoid



Ignoring fringing fields, the magnetic field inside the solenoid is constant

$$\Rightarrow \quad \Phi_{total} = N\,\Phi_{per\ turn} = N\,B_{solenoid}\,A$$

$$B_{solenoid} = \mu_0 \frac{N}{\ell} I$$

$$\Rightarrow \quad \Phi_{total} = \mu_0 \frac{N^2 A}{\ell} I$$

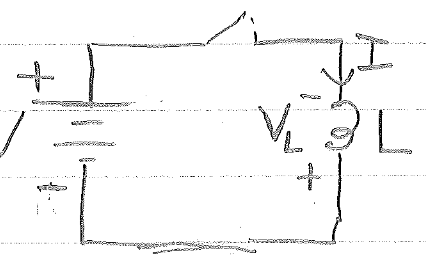
$\Rightarrow$ Inductance of a solenoid

$$\boxed{L = \mu_0 N^2 \frac{A}{\ell}}$$

Energy in Magnetic Fields

The work we do to establish a magnetic field is the work we do against the back EMF as we try to push current around a loop.

Consider the following circuit



In steady state, the work done by the battery is equal to the energy stored in the inductor

Rate @ which battery does work $\frac{dW}{dt} = IV$

$$V_L = -L\frac{dI}{dt} = -V \quad \left(\begin{array}{c}\text{kirkoff's} \\ \text{Law}\end{array}\right)$$

$$\Rightarrow \frac{dW}{dt} = LI\frac{dI}{dt} \Rightarrow W_{total} = \int_0^\infty LI\frac{dI}{dt}\,dt$$

$$\Rightarrow \boxed{W_{total} = U_{inductor} = \frac{1}{2}LI^2}$$

Analogous to $U_{capacitor} = \frac{1}{2}CV^2$

The energy stored in the inductor can be though to reside in the magnetic field

Consider the case of the solenoid

$$U = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 \frac{N^2 A I^2}{l} = \frac{1}{2\mu_0} \underbrace{\left(\mu_0 \frac{N}{l} I\right)^2}_{= |\vec{B}|^2} A l$$

$$\Rightarrow U = \frac{|\vec{B}|^2}{2\mu_0} A l$$

Energy density stored in the magnetic field $\boxed{u_B = \frac{1}{2\mu_0} |\vec{B}|^2}$ This is completely general

The energy stored in a $\vec{B}$-field = work done against the electric field associated w/ "back EMF"

Summary of energy / electric magnetic field

Work necessary to establish a charge distribution ρ	Work necessary to establish a current distribution
$U_E = \frac{1}{2} \int \rho V d^3x$	$U_B = \frac{1}{2} \int \vec{A} \cdot \vec{J} \, d^3x$
$= \frac{\epsilon_0}{2} \int \lvert\vec{E}\rvert^2 d^3x$	$= \frac{1}{2\mu_0} \int \lvert\vec{B}\rvert^2 d^3x$
$= \frac{1}{2} C V^2$	$= \frac{1}{2} L I^2$