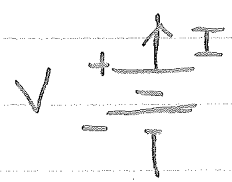
Physics 406: Lecture 5: R, L, C Circuits

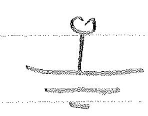
Inductors, capacitors, and resistors form the foundation of analog electronics. Let's review the fundamentals.

Circuit elements are defined by their I-V relations.

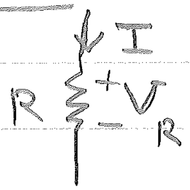
Voltage source (battery)



current runs "uphill"

Ground  $V = 0$

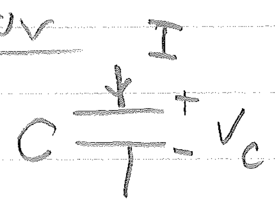
Resistor



Current runs "downhill"

$$V_R = I R$$

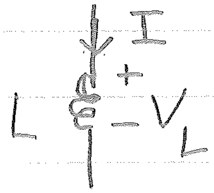
Capacitor



Current flows to charge up capacitor

$$I = C \frac{dV_C}{dt}$$

Inductor



The "back EMF" $\mathcal{E} = -V_L$

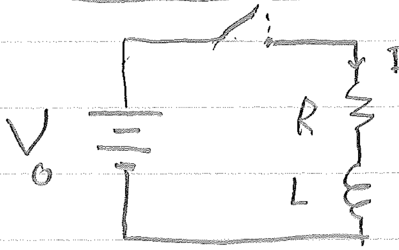
$$V_L = -\mathcal{E} = L \frac{dI}{dt}$$

Kirchoff's Law (Quasistatics)

$$\sum_{\text{closed circuit}} V = 0 \iff \oint \vec{E}\cdot d\vec{l} = 0$$

(when no flux through closed circuit)

R-L Circuit

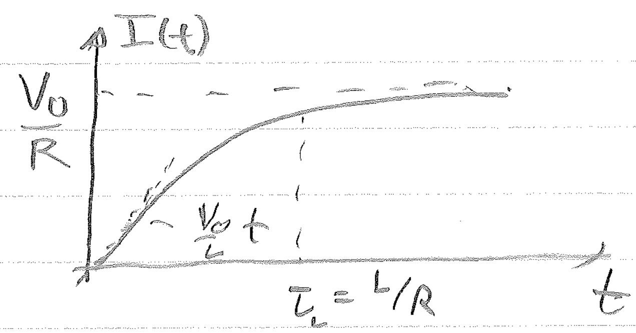


$$V_0 = V_R + V_L = IR + L\frac{dI}{dt}$$

$$\Rightarrow \frac{dI}{dt} = -\left(\frac{R}{L}\right)I + \frac{V_0}{L}$$

For short times (I small) $\frac{dI}{dt} \approx \frac{V_0}{L} \Rightarrow I \simeq \frac{V_0}{L}t$

For long times ($\frac{dI}{dt}$ small) $I \to \frac{V_0}{R}$



Full solution: $I(t) = \underbrace{Ae^{-\frac{R}{L}t}}_{\text{homogeneous solution}} + \underbrace{\frac{V_0}{R}}_{\text{particular}}$

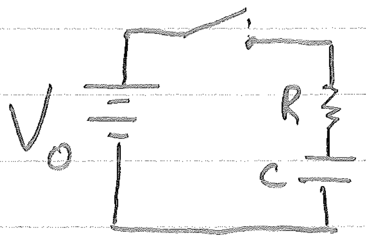
$I(0) = 0 \Rightarrow A = -\frac{V_0}{R} \Rightarrow \boxed{I(t) = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$

Inductors don't like changing current
- Acts as infinite resistor when $\frac{dI}{dt}$ large
- Acts as "short circuit" for steady current

The "time constant" for steady state behavior

$$\tau_L = L/R$$

R-C circuits



$$V_0 = V_R + V_c = IR + V_c$$

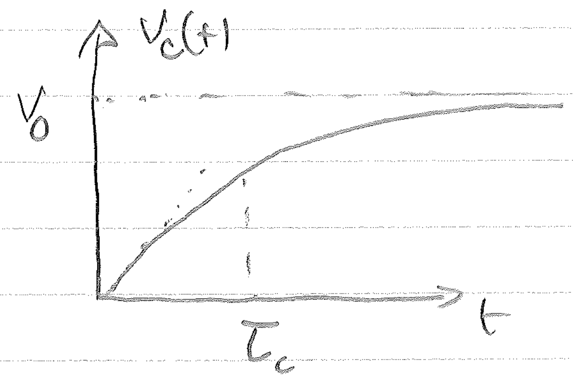
$$I = C\frac{dV_c}{dt}$$

$$\Rightarrow V_0 = RC\frac{dV_c}{dt} + V_c$$

$$\boxed{\frac{dV_c}{dt} = -\frac{1}{RC}V_c + \frac{V_0}{RC}}$$

→ Short time $V_c = \frac{t}{RC} V_0$

→ Long time $V_c = V_0$



Full solution

$$V(t) = V_0\left(1 - e^{-t/RC}\right)$$

The time constant for charging the capacitor

$$\tau_c = 1/RC$$

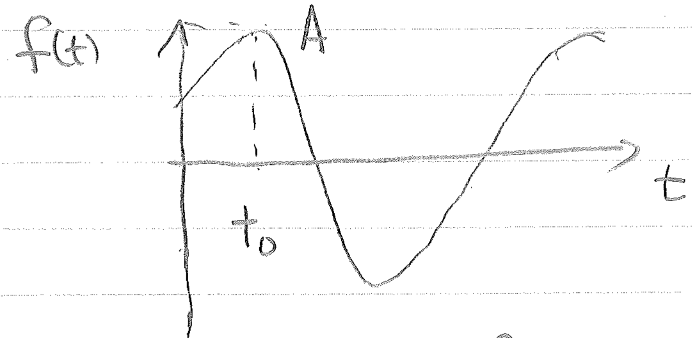
Capacitors stop flowing current when voltage stops changing $\Rightarrow$ Infinite resiter when $\frac{dV}{dt}$ small.

Alternating Current (AC)

Oscillating currents and voltage are central to physicall phenomena and electronics. Let's first establish the requisite mathematics.

Oscillating signal: period T, frequency $\nu = \frac{1}{T}$ (cycles/sec)

Hertz



$$f(t) = A\cos\left(2\pi\left(\frac{t - t_0}{T}\right)\right)$$

$$= \underset{\text{Amplitude}}{A}\cos(\omega t - \underset{\text{phase}}{\phi_0})$$

$\omega = \frac{2\pi}{T}$ = (angular freq)

By trig: $f(t) = \underbrace{A\cos\phi}_{a}\cos(\omega t) + \underbrace{A\sin\phi}_{b}\sin\omega t$

Any signal oscillating at (single) freq ω is characterized by two numbers

⇒ (• Amplitude ; A • phase : ϕ_0)

or

⇒ (• Quadratures : a (amount of $\cos\omega t$)
b (amount of $\sin\omega t$))

$$A = \sqrt{a^2+b^2}, \quad \phi = \tan^{-1}(b/a)$$

Mathematically, instead of dealing with sine and cosine as separate functions, it is more convenient (and powerful) to employ the algebra of <u>complex numbers</u> based on Euler's formula

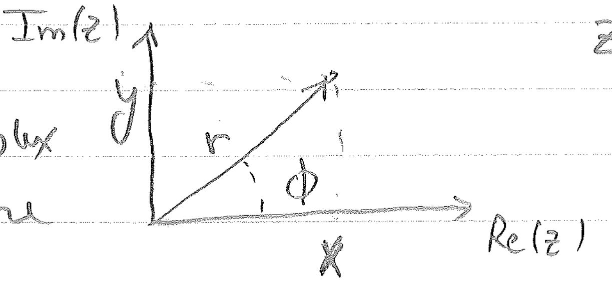
$$\boxed{e^{-i\omega t} = \cos\omega t - i\sin\omega t}$$

(minus sign is a physics convention)

Review of complex algebra

$i = \sqrt{-1}$

Complex plane



$$z = x + iy = re^{i\phi}$$

$$x = Re(z) = r\cos\phi$$

$$y = Im(z) = r\sin\phi$$

$$r = |z| = \sqrt{x^2+y^2}$$

$$\phi = \arg(z) = \tan^{-1}\left(\frac{y}{z}\right)$$

- The real/imag decomposition useful for add/sub

$$z_1 \pm z_2 = (x_1 + x_2) \pm i\,(y_1 + y_2)$$

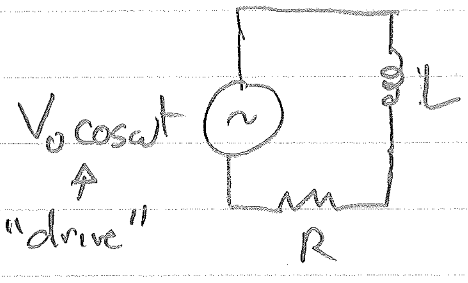
- The polar decomposition useful for mult/div (powers)

$$z_1^n z_2^m = r_1^n r_2^m \, e^{i(n\phi_1 + m\phi_2)}$$

- Important operation: Complex conjugate : $i \to -i$

$$z^* = x - iy = re^{-i\phi}, \quad |z|^2 = z^* z = x^2 + y^2$$

Application to AC Circuits



$$L \frac{dI}{dt} + IR = V_0 \cos\omega t$$

$$I(t) = I_{homogeneous} + I_{particular}$$

Homogeneous: Solution to $\frac{dI}{dt} + \frac{R}{L} I = 0$

$$\Rightarrow I_{homogenous} = I_0 e^{-R/L\, t} \quad (\text{transient})$$

Particular: Solution when transients $\Rightarrow 0$

Make ansatz $I(t) = \mathrm{Re}(\tilde{I} e^{-i\omega t})$

$\tilde{I}$ = "complex amplitude" $= |\tilde{I}| e^{i\phi_I}$

The particular solution oscillates @ freq of drive

The drive: $V_D(t) = \mathrm{Re}(V_0 e^{-i\omega t})$ (reference phase $= 0$)

Because the diff'eqn is linear we can look at both real and imaginary parts simultaneously and then take the real part in the end

$$L \underbrace{\frac{d(\tilde{I} e^{-i\omega t})}{dt}}_{= -i\omega \tilde{I} e^{-i\omega t}} + (\tilde{I} e^{-i\omega t}) R = V_0 e^{-i\omega t}$$

$\Rightarrow$ The diff'eqn becomes an algebraic eqn.

$$-i\omega \tilde{I} L + \tilde{I} R = V_0$$

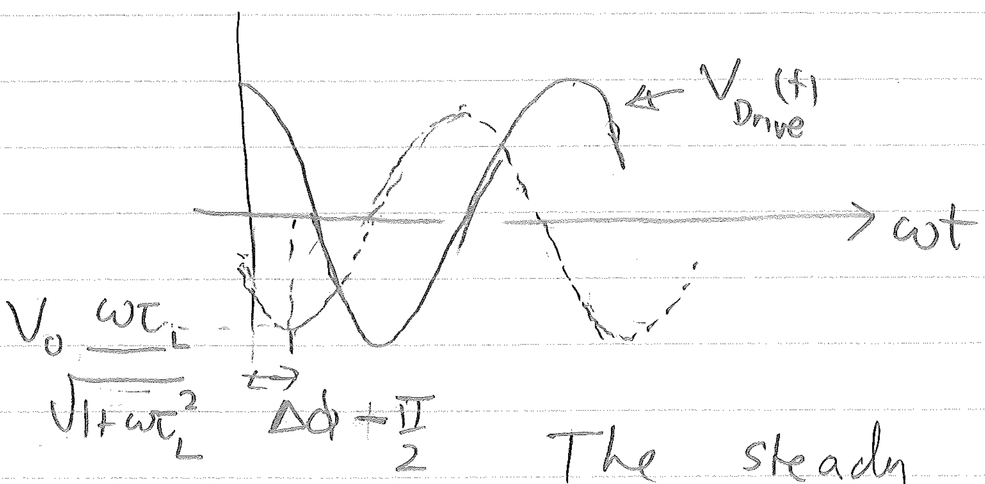
$$\Rightarrow \quad \tilde{I} = \frac{V_0}{R - i\omega L} = \frac{V_0}{\sqrt{R^2 + \omega^2 L}} e^{i\Delta\phi}$$

$$\Delta\phi = \tan^{-1}\left(\omega \frac{L}{R}\right)$$

$\Rightarrow$ Particular solution:

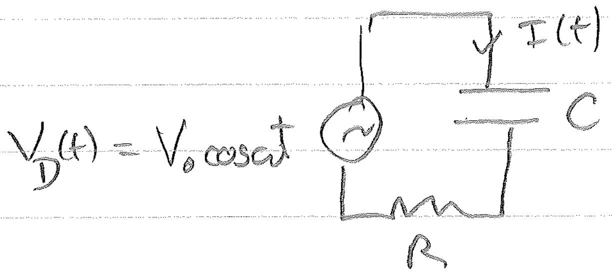
$$I_{particular} = Re\left(\tilde{I} e^{-i\omega t}\right) = \frac{V_0/R}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} \cos(\omega t - \Delta\phi)$$

$$V_{particular} = L \frac{dI}{dt} = \left(\frac{-\frac{\omega L}{R}}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}}\right) V_0 \sin(\omega t - \Delta\phi)$$



The steady state voltage across the inductor oscillates at the frequency of the drive. The signal is phase shifted due to finite response time.

AC-driven RC circuit



$$V_D(t) = V_c(t) + V_R(t)$$

$$\Rightarrow \frac{dV_D}{dt} = \frac{dV_c}{dt} + \frac{dV_R}{dt}$$

$$\Rightarrow \frac{d}{dt}(V_0 \cos\omega t) = I/C + \frac{dI}{dt} R$$

Steady state (Particular solution)

$$I(t) = \mathrm{Re}(\tilde{I} e^{-i\omega t}), \quad V_D(t) = \mathrm{Re}(V_0 e^{-i\omega t})$$

$$\Rightarrow -i\omega V_0 e^{-i\omega t} = \frac{\tilde{I}}{C} e^{-i\omega t} - i\omega \tilde{I} R e^{-i\omega t}$$

$$\Rightarrow \tilde{I} = \left(\frac{-i\omega}{-i\omega R + \frac{1}{C}}\right) V_0 = \frac{-i\omega RC}{1 - i\omega RC} V_0/R$$

$$\Rightarrow I(t) = \mathrm{Re}(\tilde{I} e^{-i\omega t}) = \frac{V_0}{R} (\omega \tau_c) \mathrm{Re}\left(\frac{-i e^{-i\omega t}}{1 - i\omega\tau_c}\right)$$

$$\tau_c = RC$$

$$\Rightarrow I(t) = \frac{V_0}{R} \frac{\omega\tau_c}{\sqrt{1 + (\omega\tau_c)^2}} \mathrm{Re}\left(-i e^{-i(\omega t - \Delta\phi)}\right)$$

$$\Delta\phi = \tan^{-1}(\omega\tau_c)$$

$$\Rightarrow I(t) = \frac{-V_0}{R} \frac{\omega\tau_c}{\sqrt{1 + (\omega\tau_c)^2}} \sin(\omega t - \Delta\phi)$$

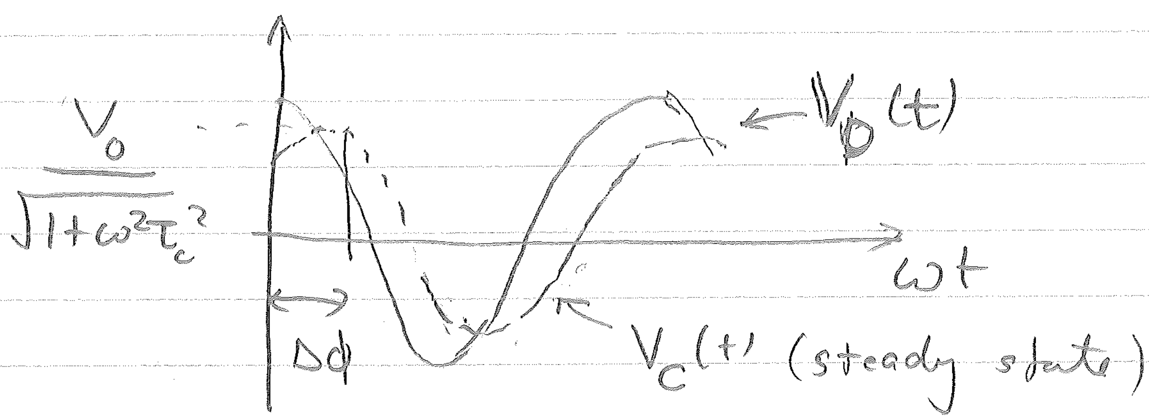
Thus the voltage drop across the capacitor

$$\frac{dV_c}{dt}(t) = \frac{I(t)}{C} \Rightarrow -i\omega \tilde{V}_c e^{-i\omega t} = \frac{\tilde{I}}{C} e^{-i\omega t}$$

$$\tilde{V}_c = \frac{\tilde{I}}{-i\omega C} = \left(\frac{1}{1-i\omega\tau_c}\right) V_0$$

$$\Rightarrow V_c(t) = \mathrm{Re}\left(\tilde{V}_c e^{-i\omega t}\right) = \mathrm{Re}\left(\frac{1}{1-i\omega\tau_c} V_0 e^{-i\omega t}\right)$$

$$V_c(t) = \frac{V_0}{\sqrt{1+(\omega\tau_c)^2}} \cos(\omega t - \Delta\phi)$$



Voltage across the capacitor is also phase shifted in steady-state