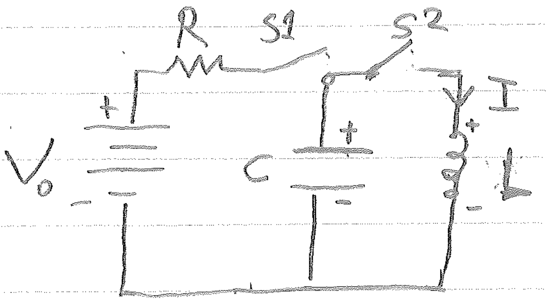
Physics 406: Lecture 6: LC- Oscillators

Consider the following circuit:



Initially switch S1 is closed and S2 is open. After the capacitor capacitor is fully charged, S1 is opened as S2 is closed current starts flowing from the capacitor. How does the current evolve in time?

Physically we expect the following. As the capacitor starts to discharge and current flows from C to L, the inductor sets up a back EMF to oppose the changing current. Eventually, when the charge is depleted, the current stops changing. At that point there is no back EMF and the capacitor starts charging up again until the current decreases back to zero, and the whole cycle repeats.

Thus we expect an <u>oscillation</u> of the current and voltage.

Mathematically:

$$V_L = L\frac{dI_L}{dt} \qquad I_L = -I_C = -C\frac{dV_C}{dt}$$

$$V_C = V_L \Rightarrow I_L = -C\frac{dV_L}{dt}$$

$$\therefore \quad V_L = -LC\frac{d^2V_L}{dt} \Rightarrow \boxed{\frac{d^2V_L}{dt} + \frac{1}{LC}V_L = 0}$$

This is the different equation of a simple harmonic oscillator (SHO) $\frac{d^2x}{dt} + \omega_0^2 x = 0$

The natural resonance frequency

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

Solution $V_L(t) = V_c(t) = a\cos\omega_0 t + b\sin\omega_0 t$

$$I_L(t) = -I_c(t) = C\frac{dV_L}{dt}$$

$$= \frac{a}{\sqrt{L/C}}\sin\omega_0 t - \frac{b}{\sqrt{L/C}}\cos\omega_0 t$$

Initial conditions

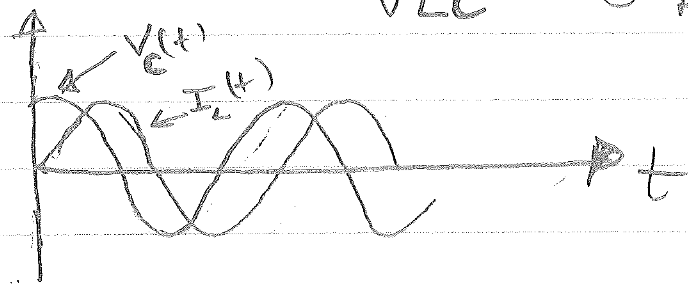
$$V_L(0) = V_0, \quad I_L(0) = 0 \Rightarrow \begin{matrix} a = V_0 \\ b = 0 \end{matrix}$$

$$= V_c(t)$$

$$\Rightarrow V_L(t) = V_0 \cos\omega_0 t, \quad I_L(t) = I_0 \sin\omega_0 t$$

where $V_0/I_0 = \sqrt{L/C}$ (characteristic impedance)

$\omega_0 = \dfrac{1}{\sqrt{LC}}$ (resonance frequency)



The voltage across the capacitor is 90° out of phase with the current through the inductor.

Energy stored in the inductor:

$$U_L = \frac{1}{2} L I_L^2 = \frac{1}{2} L I_0^2 \sin^2\omega_0 t$$

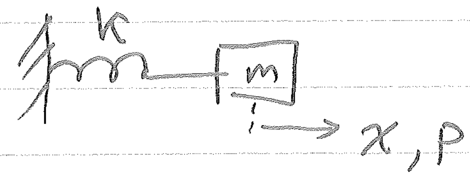
Energy stored in the capacitor

$$U_C = \frac{1}{2} C V_c^2 = \frac{1}{2} C V_0^2 \cos^2\omega_0 t$$

$$= \frac{1}{2} L I_0^2 \cos^2\omega_0 t$$

The energy is oscillating back and forth between the inductor (magnetic energy) and the capacitor (electric energy)

The current and voltage in an LC-oscillator are analogous to the position and momentum of a mass on a spring S.H.O



$$\dot{X} = P/m \iff \dot{I} = V/L$$

$$\dot{p} = -kX \iff \dot{V} = -\frac{1}{C} I$$

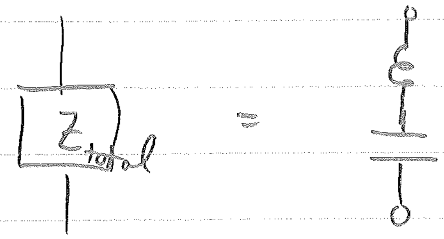
Inductor like inertia; Capacitor like restoring force

In the mechanically case, energy oscillates between kinectic energy of motion and potential energy of the spring. A change in position drives momentum, but the restoring force reduces the momentum proportial to the position

Aside: A quantization of a circuit follows with I, V like canonical $\hat{X}, \hat{P}$ that don't commute!

Complex Impedance

To treat an LC circuit driven by an AC drive we turn to complex impedance



series LC element

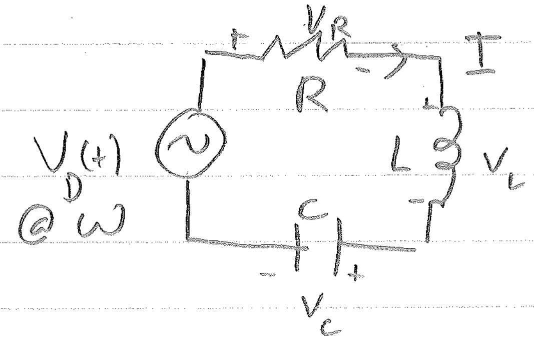
$$Z_{total} = Z_c + Z_L = \frac{1}{-i\omega C} - i\omega L$$

$$\boxed{Z_{total}(\omega) = \frac{1-\omega^2 LC}{-i\omega C} = \frac{1-\omega^2/\omega_0^2}{-i\omega C}}$$

Note: when $\omega = \omega_0$ $Z_{total} = 0$

The out of phase impedances cancel.

RLC Circuit



$$V_D(t) = V_R + V_L + V_C$$

$$I_R = I_L = I_C \equiv I(t)$$

$$I(t) = I_{Homogenous} + I_{particular}$$

transient   steady-state

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Transient solution $V_D = 0$

$$\Rightarrow V_R + V_R + V_c = 0$$

$$IR + L\frac{dI}{dt} + \frac{Q}{c} = 0$$

$$R\frac{dI}{dt} + L\frac{d^2I}{dt} + \frac{1}{c}I = 0$$

$$\Rightarrow \boxed{\begin{aligned} &\ddot{I} + \Gamma\dot{I} + \omega_0^2 I = 0 \\ &(\text{where } \omega_0 = 1/\sqrt{LC}, \quad \Gamma = L/R) \end{aligned}} \leftarrow \text{Damped } \underline{\text{SHO}}$$

Ansatz: $I = \mathrm{Re}(\tilde{I}e^{st})$ $\quad$ <u>s complex</u>

$$\Rightarrow (s^2 + s\Gamma + \omega_0^2)\tilde{I} = 0$$

$$\Rightarrow s^2 + s\Gamma + \omega_0^2 = 0 \quad \text{(quadratic eqn)}$$

Two roots: $s_\pm = -\frac{\Gamma}{2} \pm \frac{\sqrt{\Gamma^2 - 4\omega_0^2}}{2}$

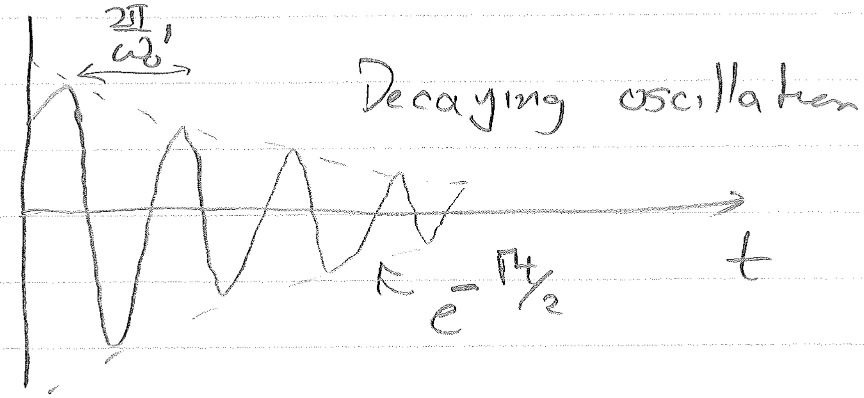
Underdamped $\omega_0 > \frac{\Gamma}{2} \Rightarrow s_\pm = \pm i\omega_0' - \frac{\Gamma}{2}$

where $\omega_0' = \sqrt{\omega_0^2 - \Gamma^2/4}$

$$\Rightarrow I(t) = \mathrm{Re}(\tilde{I}\underbrace{e^{-\Gamma t/2}}_{\text{decay}} \underbrace{e^{\pm i\omega_0' t}}_{\text{oscillation}})$$

$$= Ae^{-\Gamma t/2}\cos(\omega_0' t - \phi)$$

Transient solution



Steady state solution (for $t \gg \frac{1}{\Gamma}$)

In steady state, the current oscillated at the frequency of the drive. $V_D(t) = Re(V_0 e^{-i\omega t})$

$$I(t) = Re(\tilde{I} e^{-i\omega t}) \qquad \tilde{I} = \frac{V_0}{Z_{total}}$$

$$\Rightarrow Z_{total} = Z_R + Z_C + Z_L$$

$$= R + \frac{1}{-i\omega C} + i\omega L = R + i\left(\frac{1}{\omega C} - \omega L\right)$$

$$\therefore \boxed{I(t) = Re\left(\frac{V_0 e^{-i\omega t}}{Z_{total}}\right) = \frac{V_0}{|Z_{tot}|}\cos(\omega t - \phi)}$$

where $|Z_{total}| = \sqrt{R^2 + \frac{1}{\omega^2 C^2}\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}$

$$\phi = \tan^{-1}\left(\frac{1}{\omega RC} - \frac{\omega L}{R}\right)$$

Voltage across the capacitor $\quad V_c = Re(\tilde{V}_c e^{-i\omega t})$

$$\tilde{V}_c = \tilde{I} Z_c = \left(\frac{V_0}{Z_{tot}}\right) Z_c$$

$$\therefore \frac{\tilde{V}_c}{V_0} = \frac{Z_c}{Z_{tot}} \equiv F(\omega) \quad (\text{"response function"})$$

$$F(\omega) = \frac{\frac{i}{\omega C}}{R + i\left(\frac{1}{\omega C} - \omega L\right)} = \frac{1}{-i\omega RC - \omega^2 LC + 1}$$

$$\Rightarrow \boxed{F(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}}$$

Resance $\quad \omega = \omega_0, \quad F \to \infty$ as $\Gamma \to 0$

Let's examine the resonant behavior with $\Gamma \ll \omega_0$

Look near $\omega \approx \omega_0 \quad$ let $\Delta \equiv \omega - \omega_0$

$\Gamma, \Delta \ll \omega, \omega_0$

$$\Rightarrow \omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega) = (-\Delta)(\Delta + 2\omega_0)$$

$$\approx -2\omega_0 \Delta, \quad \omega\Gamma \approx \omega_0 \Gamma$$

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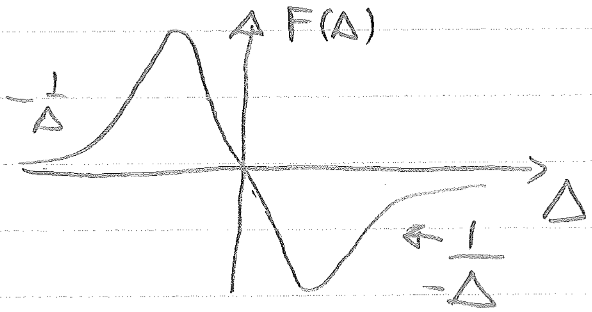
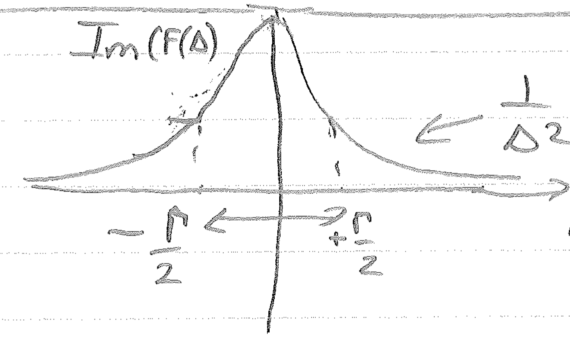
Near resonant response funtion

$$\boxed{F(\Delta) \approx \frac{\omega_0/2}{-\Delta - i\frac{\Gamma}{2}}} \qquad (\Delta = \omega - \omega_0)$$

$$\boxed{Re\left(F(\omega)\right) = \frac{\omega_0}{2}\left(\frac{-\Delta}{\Delta^2 + \frac{\Gamma^2}{4}}\right), \quad Im\left(F(\omega)\right) = \frac{\omega_0}{2}\left(\frac{\frac{\Gamma}{4}}{\Delta^2 + \frac{\Gamma^2}{4}}\right)}$$





Lorentzian

Γ = Full width half max

$$V_c(t) = Re\left(F(\omega)\, V_0\, e^{-i\omega t}\right)$$

$$= \underbrace{Re(F(\omega))\, V_0 \cos\omega t}_{\text{in plase with drive}} + \underbrace{Im(F(\omega))\, V_0 \sin\omega t}_{\text{in quadrature with drive}}$$

$$V_c(t) = |F(\omega)|\, V_0 \cos\left(\omega t - \arg(F(\omega))\right)$$

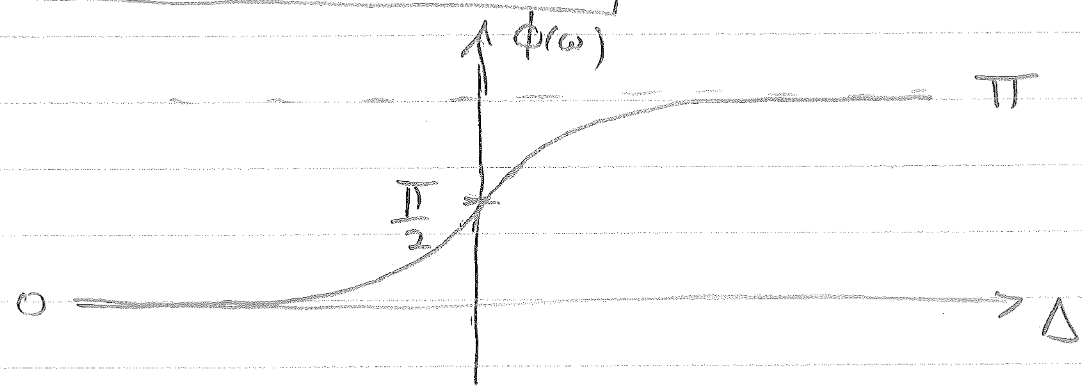
$$= \frac{\frac{\omega_0}{2}}{\sqrt{\Delta^2 + \frac{\Gamma^2}{4}}}\, V_0 \cos\left(\omega t - \arg(F(\omega))\right)$$

- The amplitude of V_C is maximum on resonance $(\Delta = 0)$

- The phase shift

$$\phi(\omega) = \arg(F(\omega)) = -\arg(Z_{total})$$

$$\boxed{\phi(\omega) = -\tan^{-1}\left(\frac{\Gamma}{2\Delta}\right)}$$ Phase shift



- Well below resonance, V_c oscillates in phase with $V_D(t)$

- On resonance, V_C oscillates 90° out of phase with V_D

- Well above resonance, V_c oscillates 180° out of phase with V_D