Physics 406: Lecture 4

Maxwell's Equations & Waves

Up to this point we have the following field eqns.

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

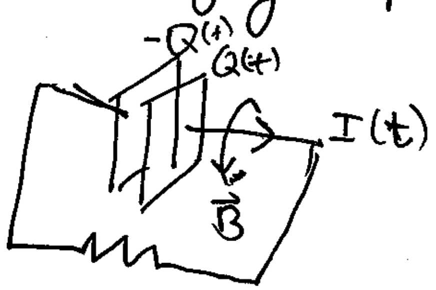
$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

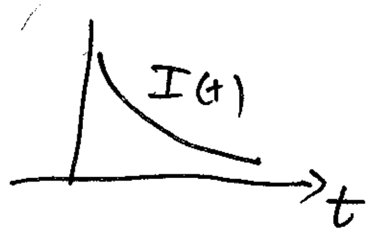
These are all "empirical laws". Maxwell changed everything from theoretical reasoning to make a new prediction.

Problem with Ampère's Law:
Only hold for steady currents

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0 \vec{\nabla}\cdot\vec{J} = 0$$

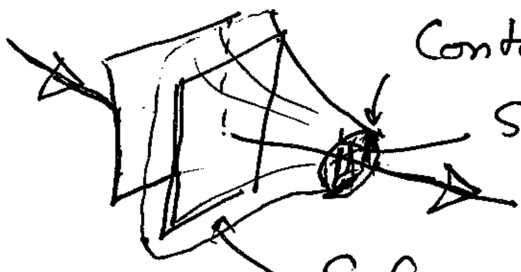
E.g.: Discharging capacitor



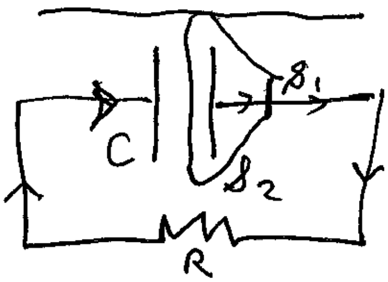


not steady

Inconsistency in Stokes's theorem



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int_{S_1} \vec{J}\cdot d\vec{a}_1 = \mu_0 \int_{S_2} \vec{J}\cdot d\vec{a}_2$$



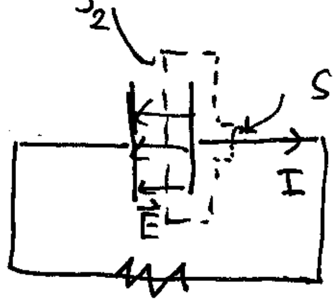
But, no current is flowing through S_2 ?

$\Rightarrow$ Something else must be the source of $\vec{B}$-field for non-steady currents

Need an addition source term:

Charge Conservation $\Rightarrow$ $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \vec{\nabla}\cdot\vec{E})$

$$\Rightarrow \vec{\nabla}\cdot\Big(\vec{J} + \underbrace{\epsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\equiv \vec{J}_d \text{ “Displacement current”}}\Big) = 0$$



$$|\vec{E}| = \frac{\sigma(t)}{\epsilon_0} = \frac{Q(t)}{A\epsilon_0}$$

$$\Rightarrow \left|\frac{\partial \vec{E}}{\partial t}\right| = \frac{I(t)}{A\epsilon_0}$$

$$\int \vec{J}_d \cdot da_2 = \int_{S_2} \epsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}_2 = \left(\frac{I}{A}\right) A = I = \int_{S_1} \vec{J} \cdot da_1 \checkmark$$

So charge conservation gives us exactly the necessary ingredient to make everything self-consistent. Yay!

With the extra "displacement current" we have now the complete Maxwell's Eqns:

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 && \vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} && \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \end{aligned}}$$

This, together with Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

defines all of electro-magnetism!

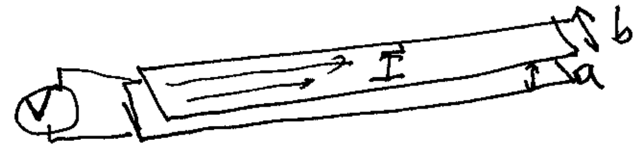
Mawell's Great Prediction

Consider Maxwell's Equations in free space: $\rho(\vec{r}) = J(\vec{r}) = 0$

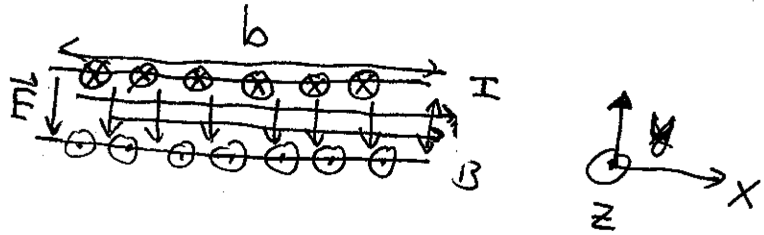
$$\Rightarrow \boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = 0 & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\dfrac{\partial\vec{E}}{\partial t} \end{array}}$$

Set of first order Coupled PDE'S

~~Elmag~~ Example: Parralel Plates



Edge on:



$$\vec{E} = E_y(z)\,\hat{y}, \quad \vec{B} = B_x(z)\,\hat{x}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_y}{\partial y} = 0, \quad \vec{\nabla}\cdot\vec{B} = \frac{\partial B_x}{\partial x} = 0$$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = -\hat{x}\frac{\partial E_y}{\partial z} = -\hat{x}\frac{\partial B_x}{\partial t} \Rightarrow \boxed{\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}}$$

$$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & 0 & 0 \end{vmatrix} = \hat{y}\frac{\partial B_x}{\partial z} = \hat{y}\mu_0\epsilon_0\frac{\partial E_y}{\partial t} \Rightarrow \boxed{\frac{\partial B_x}{\partial z} = \mu_0\epsilon_0\frac{\partial E_y}{\partial t}}$$

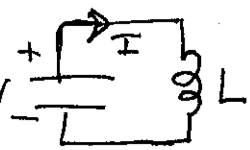
Note $E_z(z,t) \propto V(z,t)$ (local voltage at z)

$B_x(z,t) \propto I(z,t)$ (local current at z)

$$\Rightarrow \quad \frac{\partial V}{\partial z} = \underset{\substack{\uparrow \\ \text{Capacitance/length}}}{\mathcal{C}} \frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial z} = \underset{\substack{\uparrow \\ \text{Inductance/length}}}{\mathcal{L}} \frac{\partial V}{\partial t}$$

(see Problem Set)

Compare w/ L-C oscillator



$$\left.\begin{aligned} I &= -C\frac{dV}{dt} \\ V &= L\frac{dI}{dt} \end{aligned}\right\} \text{Local exchange of energy}$$

"Transmission line" $V(z,t)$, $I(z,t)$ not <u>instaneously</u> related to V + I at different position: <u>Finite Propagation speed</u>. Same is true of <u>fields</u>

$$\boxed{\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}} \qquad \boxed{\frac{\partial B_x}{\partial z} = \mu_0\epsilon_0 \frac{\partial E_y}{\partial t}}$$

$$\Longrightarrow \quad \frac{\partial^2 E_y}{\partial z^2} = \frac{\partial}{\partial t}\left(\frac{\partial B_x}{\partial z}\right) = \mu_0\epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{\partial^2 B_x}{\partial z^2} = \frac{\partial}{\partial t}\left(\mu_0\epsilon_0 \frac{\partial E_y}{\partial z}\right) = \mu_0\epsilon_0 \frac{\partial^2 B_x}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial z^2} - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right)\begin{pmatrix} E_y \\ B_x \end{pmatrix} = 0}$$

$\boxed{\vec{E} \text{ and } \vec{B} \text{ satisfy the wave equation}}$

Maxwell's great prediction

Changing $\vec{E}$ generates changing $\vec{B}$ which generates changing $\vec{E}$ which " " " and so on...

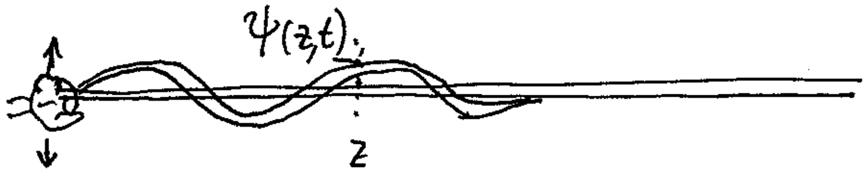
$\Rightarrow$ Propagating fields!

Wave equation (propagation along z-axis)

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi(z,t) = 0$$

Wave amplitude

Example: Wave on string



$f(t) = \psi(0,t)$: Ansatz: $\psi(z,t) = f\left(t - \frac{z}{v}\right)$

$\equiv t_{ret}$ (retarded time)

$$\frac{\partial \psi(z,t)}{\partial t} = \frac{\partial t_r}{\partial t}\frac{\partial f}{\partial t_r} = \frac{\partial f}{\partial t_r}$$

$$\frac{\partial \psi(z,t)}{\partial z} = \frac{\partial t_r}{\partial z}\frac{\partial f}{\partial t_r} = -\frac{1}{v}\frac{\partial f}{\partial t_r}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 f}{\partial t_r^2} , \quad \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t_r^2}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \checkmark$$

$\Rightarrow$ v is the progation velocity

From Maxwell eqns: $v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$\mu_0 = 4 \times 10^{-7} \frac{N}{A^2}$ $\quad \epsilon_0 = 8.85 \times 10^{-12} \frac{A^2 s^2}{N m^2}$

$$\Rightarrow \boxed{v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8 \, m/s = c}$$

speed of light

Tremendous triumph for physics!

Empirical laws (Faraday, Coulomb, Ampère) combined with theorical insight $\Rightarrow$ New physical prediction

- Light is an electromagnetic wave
- Physical foundation of optics!

Review of basic wave concepts

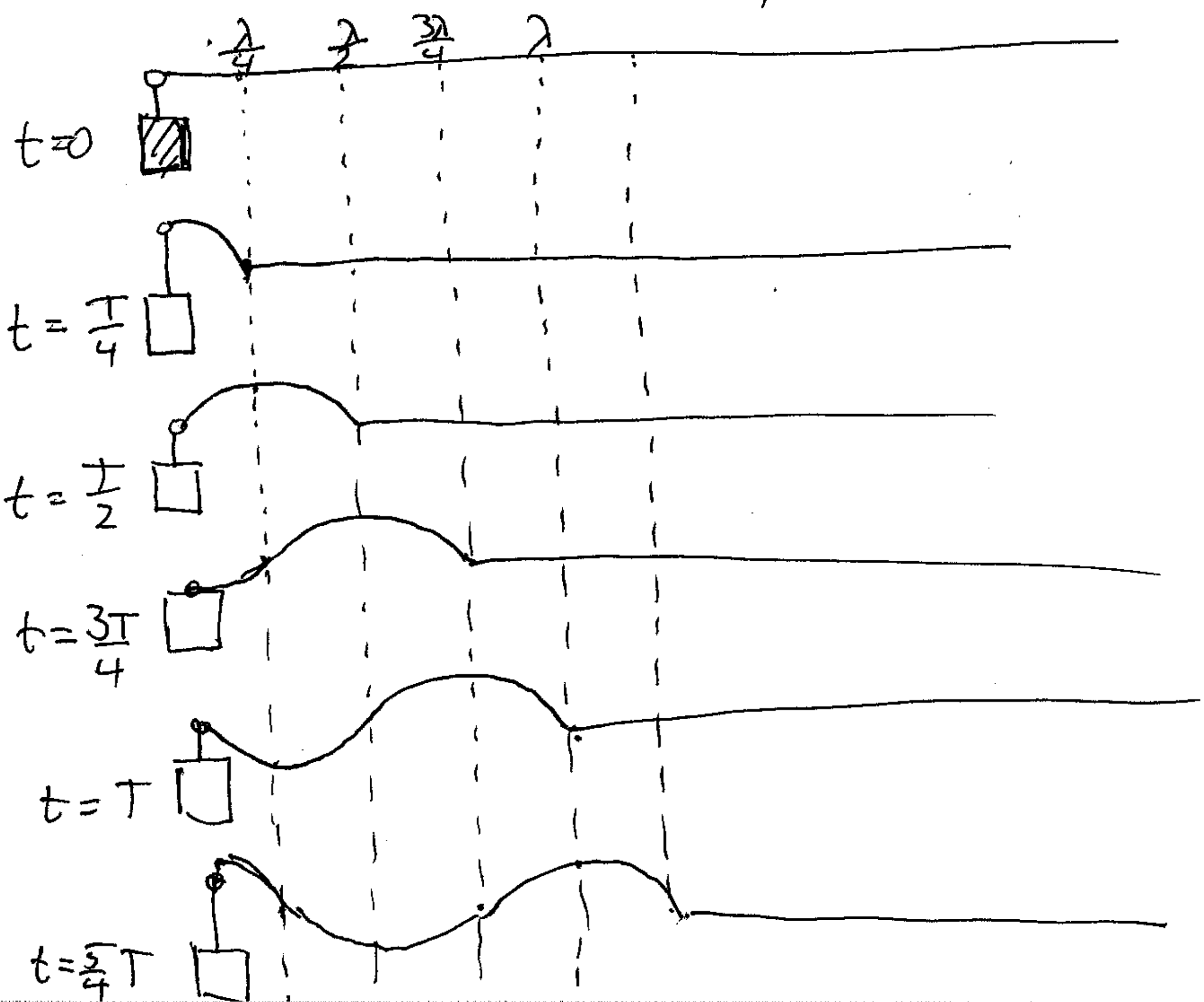
Harmonic signal propagation: $f(t) = A\cos(\omega t - \phi)$

$$\Rightarrow \psi(z,t) = A\cos(\omega t_{ret} - \phi)$$
$$= A\cos(\omega t - \tfrac{\omega}{v} z - \phi)$$
$$= A\cos(\omega t - kz - \phi)$$

$$\Rightarrow \boxed{\psi(z,t) = A\cos(kz - \omega t + \phi)}$$ ← Harmonic wave

$$k = \frac{\omega}{v} \equiv \text{"wave number"}$$

Example: $f(t) = A\sin\omega t$, $\psi(z,t) = A\sin(kz - \omega t)$



At $t=T$ $\quad \psi(z,T) = A\sin(kz - 2\pi) = A\sin kz$

Periodic in space: spatial period $\equiv \boxed{\lambda = \frac{2\pi}{k}}$

λ = wave length

$$\boxed{\lambda = \frac{2\pi}{k} = 2\pi \frac{v}{\omega} = \frac{v}{\nu}} \qquad \boxed{\left(\nu = \frac{1}{T}\right)}$$

v = "phase velocity" : Velocity @ which a point of constant phase propagates in a harmonic wave.

$\psi(z,t) = A\sin(\phi(z,t))$

phase of oscillation as a function of z and t

$\phi(z,t) = kz - \omega t \Rightarrow d\phi = k\,dz - \omega dt$

$$\Rightarrow \boxed{v_\phi = \left.\frac{dz}{dt}\right|_{d\phi=0} = \frac{\omega}{k}}$$

Electromagnetic Spectrum

$c = 3\times 10^8$ m/s for all frequencies

$\lambda = \frac{c}{\nu}$: Wavelength for different frequencies

	Radio/TV	micro wave	infra red	visible	ultra violet	x ray	gamma ray
ν (Hz)	$10^3 - 10^9$	$10^9 - 10^{12}$	$10^{12} - 10^{15}$	10^{15}	$10^{15} - 10^{18}$	$10^{18} - 10^{20}$	$10^{20} \rightarrow$
λ (m)	$10^5 - 1$	$10^{-1} - 10^{-3}$	$10^{-3} - 10^{-6}$	10^{-6}	$10^{-6} - 10^{-9}$	$10^{-9} - 10^{-11}$	$10^{-11} \rightarrow$

Relationship between ω and k : "Dispersion relation"

$$\boxed{\omega = ck}$$

$\Uparrow$

linear relationship in free space

$\Rightarrow$ $v = \frac{\omega}{k} = c$ same $\forall$ freq.

We will see that inside materials we can have "nonlinear" dispersion relation

$\Rightarrow$ Dispersion (like in a prism)