Physics 406: Lecture 8

Plane Wave Propagation I

Consider Maxwell's Eqns in vacuum ($\rho = \vec{J} = 0$)

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = 0 & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t} \end{array}}$$

Set of coupled, first order, ODE's

Decouple by taking second derivative

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{E}}) - \nabla^2\vec{E} \quad \text{(vector identity)}$$

$$= \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t^2}$$

$$\Rightarrow \boxed{\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}\vec{E} = 0}$$

sometime written as $\Box\vec{E} = 0$ $\quad \Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

Dalambertian

Similarly $\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{B})} = \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$

$$= \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{B}}) - \nabla^2\vec{B}$$

$$\Rightarrow \boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0}$$

$$\boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0}$$

Three dimensional wave equation for each vector component of $\vec{E}$ & $\vec{B}$

(Six second-order PDE's)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E_x(\vec{r},t) = 0 \quad \text{etc.}$$

Harmonic Waves in 3D

We seek solutions which oscillate like ωt at any point in space. We thus write the ansatz

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left(\vec{\underset{\sim}{E}}(\vec{r})\, e^{-i\omega t}\right) \qquad \vec{B}(\vec{r},t) = \mathrm{Re}\left(\vec{\underset{\sim}{B}}(\vec{r})\, e^{-i\omega t}\right)$$

complex amplitude (vector) @ $\vec{r}$

Work with complex quantities, then take real part in end.

$$\frac{\partial}{\partial t} \Longrightarrow -i\omega t$$

(time derivative on $e^{-i\omega t}$ is multiplication by $-i\omega$)

Maxwell's Eqns. for Harmonic Time dependence

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{\underset{\sim}{E}} = 0 & \vec{\nabla}\cdot\vec{\underset{\sim}{B}} = 0 \\[2ex] \vec{\nabla}\times\vec{\underset{\sim}{E}} = i\omega\vec{\underset{\sim}{B}} & \vec{\nabla}\times\vec{\underset{\sim}{B}} = -i\dfrac{\omega}{c^2}\vec{\underset{\sim}{E}} \end{array}}$$

Wave equation w/ harmonic time dependence

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\begin{pmatrix}\vec{\underset{\sim}{E}} \\ \vec{\underset{\sim}{B}}\end{pmatrix} = 0 \quad \text{(Helmholtz eqn.)}$$

Plane wave solution:

Take spatially periodic ansatz (in 3D)

$$\vec{\underset{\sim}{E}}(\vec{r}) = \vec{\underset{\sim}{E}}_0 e^{i\vec{k}\cdot\vec{r}} \qquad \vec{\underset{\sim}{B}}(\vec{r}) = \vec{\underset{\sim}{B}}_0 e^{i\vec{k}\cdot\vec{r}}$$

$\vec{k} \equiv$ "wave vector"

Aside

$$\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) e^{i(k_x x + k_y y + k_z z)}$$

$$= \hat{x}\left(ik_x e^{i\vec{k}\cdot\vec{r}}\right) + \hat{y}\left(ik_y e^{i\vec{k}\cdot\vec{r}}\right) + \hat{z}\left(ik_z e^{i\vec{k}\cdot\vec{r}}\right)$$

$$= i\,(k_x\hat{x} + k_y\hat{y} + k_z\hat{z})\, e^{i\vec{k}\cdot\vec{r}}$$

$$= i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$$

$\Rightarrow \quad \vec{\nabla} \Leftrightarrow i\vec{k}$ when acting on spatially harmonic function $e^{i\vec{k}\cdot\vec{r}}$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} \Leftrightarrow -\vec{k}\cdot\vec{k} = -|\vec{k}|^2$$

$\Rightarrow$ Solution to wave equation in 3D

$$\left(-|\vec{k}|^2 + \frac{\omega^2}{c^2}\right)\begin{pmatrix}\tilde{\vec{E}} \\ \tilde{\vec{B}}\end{pmatrix} = 0$$

$$\Rightarrow \quad \boxed{|\vec{k}| = \frac{\omega}{c}}$$

$$\Rightarrow \quad \boxed{\vec{E} = \mathrm{Re}\left(\tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right) \qquad \vec{B} = \mathrm{Re}\left(\tilde{\vec{B}}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right)}$$

Interpretation

Suppose $\vec{k} = k\hat{z}$ $\qquad \tilde{\vec{E}}_0 = \vec{E}_0 e^{i\phi}$

$$\Rightarrow \quad \vec{E} = \vec{E}_0 \cos(kz - \omega t + \phi)$$

$\omega = \frac{2\pi}{T}$ $\qquad k = \frac{2\pi}{\lambda}$ $\qquad \lambda =$ wave length

= "wave number"

Wave propagating in z-direction

$\Longrightarrow$ Direction of $\vec{k}$ is direction of propagation.

Attention:

Suppose $\vec{E} = \vec{E}_0 \cos(kz + \omega t + \phi)$; Direction of propagation?

Note: $\vec{E} = \vec{E}_0 \cos(-kz - \omega t - \phi) \quad \Rightarrow \quad \vec{k} = -k\hat{z}$

$\Longrightarrow$ propagates in $-z$ direction

Important relationships for plane waves

$$\vec{E} = \mathrm{Re}\left(\tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right)$$
$$\vec{B} = \mathrm{Re}\left(\tilde{\vec{B}}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right)$$

are solutions to 3D wave equation

The parameters $\tilde{\vec{E}}_0$, $\tilde{\vec{B}}_0$, $\vec{k}$, ω are <u>constrained</u> by Maxwell's eqns.

- $|\vec{k}| = \frac{\omega}{c}$ (We will write $\vec{k} = \frac{\omega}{c}\hat{k}$)
- $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow i\vec{k}\cdot\tilde{\vec{E}}_0 = 0 \Rightarrow \boxed{\tilde{\vec{E}}_0 \perp \vec{k}}$
- $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow i\vec{k}\cdot\tilde{\vec{B}}_0 = 0 \Rightarrow \boxed{\tilde{\vec{B}}_0 \perp \vec{k}}$

$\Rightarrow$ In vacuum EM are <u>transverse</u>

(Direction of fields $\perp$ to direction of propagation)

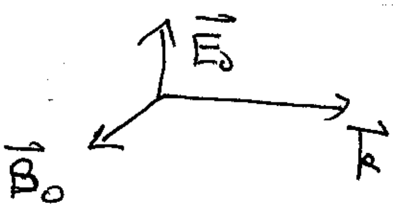
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow i\vec{k}\times\tilde{\vec{E}}_0 = i\omega\tilde{\vec{B}}_0$$

$$\Rightarrow \boxed{\hat{k}\times\tilde{\vec{E}}_0 = \frac{\omega}{k}\tilde{\vec{B}}_0 = c\,\tilde{\vec{B}}_0}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow i\vec{k}\times\tilde{\vec{B}}_0 = -i\frac{\omega}{c^2}\vec{E}_0$$

$$\Rightarrow \boxed{\hat{k}\times\vec{B}_0 = -\frac{\omega}{kc^2}\tilde{\vec{E}}_0 = -\frac{1}{c}\tilde{\vec{E}}_0}$$

$$\Rightarrow \boxed{|\tilde{\vec{E}}_0| = c\,|\tilde{\vec{B}}_0|} \qquad \tilde{\vec{E}}_0 \perp \tilde{\vec{B}}_0 \perp \vec{k}$$



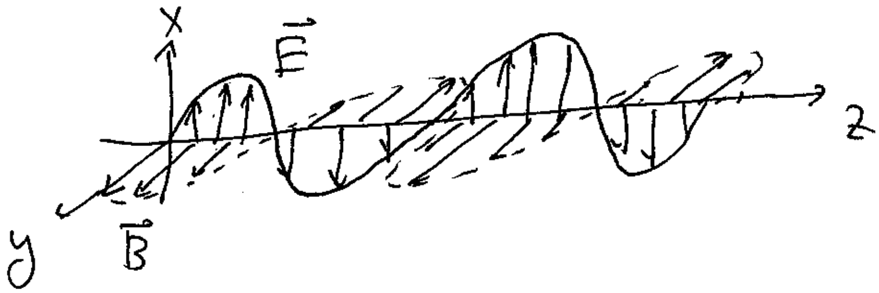
Example:

$$\vec{E}(\vec{r}) = \hat{x}\, E_0 \cos(kz - \omega t) \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$\vec{B}(\vec{r}) = \hat{y}\, \frac{E_0}{c} \cos(kz - \omega t) \qquad \vec{k} = \frac{\omega}{c}\hat{z}$$

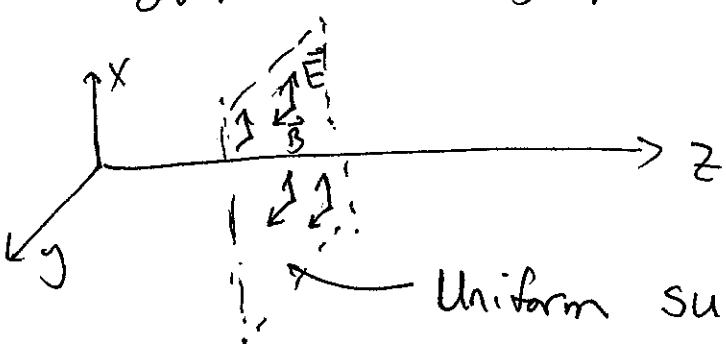
Checks: $\vec{k}\cdot\vec{E} = \vec{k}\cdot\vec{B} = 0$ ✓

Direction of $\vec{E}\times\vec{B} = \hat{x}\times\hat{y} = \hat{z}$ = direction of $\vec{k}$ ✓



"Snap-shot" of EM wave @ some time

At a given time $\vec{E}$ + $\vec{B}$ are <u>uniform</u> over the $\hat{x}$-$\hat{y}$ plane



Uniform surface of constant $\vec{E}, \vec{B}$
"wave front"

$\Rightarrow$ <u>Plane Wave</u>

More generally: Let $\vec{k} = \hat{k}\frac{\omega}{c}$ $\quad \vec{E} = \hat{\varepsilon} E_0$

$$\Rightarrow \vec{E} = \vec{\varepsilon}\, E_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$\vec{B} = (\hat{k}\times\vec{\varepsilon})\, \frac{E_0}{c} \cos(\vec{k}\cdot\vec{r} - \omega t)$$

Let $\phi(\vec{r}, t) = \vec{k}\cdot\vec{r} - \omega t$

$=$ Plane of oscillation as a function of space and time.

@ some fixed time t_0 $\quad \phi(\vec{r}) = \vec{k}\cdot\vec{r} - \phi_0$

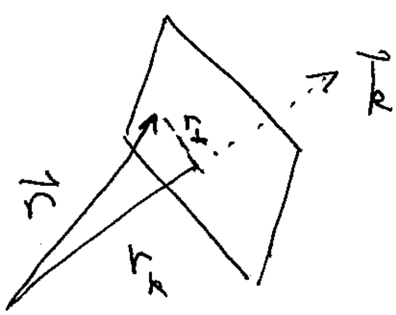
where $\phi_0 = \omega t_0$

What is the surface of constant phase?

Let $\vec{r} = r_k \hat{k} + r_\perp \hat{k}_\perp$

$\Rightarrow$ On a plane $\perp$ to $\hat{k}$

$\vec{k}\cdot\vec{r} = k r_k = \text{Constant}$



$\Rightarrow$ $\boxed{\text{Surfaces of constant phase are planes } \perp \text{ to } k}$

$\Rightarrow$ Plane Wave.