Physics 406

Lecture 9: Plane Wave Propagation (II)

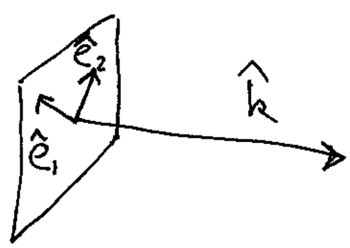
Polarization

Given a plane wave propagating in the $\hat{k}$-direction, the most general electric field is

$$\vec{E}(\vec{r},t) = E_1 \cos(\vec{k}\cdot\vec{r} - \omega t + \phi_1)\,\hat{e}_1 + E_2 \cos(\vec{k}\cdot\vec{r} - \omega t + \phi_2)\,\hat{e}_2$$

Where $\hat{e}_1$ and $\hat{e}_2$ are two vectors that form a basis for the plane $\perp$ to $\hat{k}$ (usually chosen orthonormal)

$$\hat{e}_1\cdot\hat{e}_2 = 0 \qquad \hat{e}_1\cdot\hat{e}_1 = \hat{e}_2\cdot\hat{e}_2 = 0$$



Using the complex amplitude notation

$$\vec{E}(\vec{r},t) = \mathrm{Re}\Big(E_1 e^{i\phi_1}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{e}_1 + E_2 e^{i\phi_2}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{e}_2\Big)$$

Define $E_0 = \sqrt{E_1^2 + E_2^2}$ $\quad a_1 \equiv \dfrac{E_1}{E_0}$ $\quad a_2 \equiv \dfrac{E_2}{E_0}$

$$\Rightarrow \vec{E}(\vec{r},t) = \mathrm{Re}\Big(E_0 e^{i\phi_1}\hat{\epsilon}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\Big)$$

where $\boxed{\hat{\epsilon} = a_1\hat{e}_1 + e^{i\chi} a_2 \hat{e}_2}$ $\qquad \boxed{\chi = \phi_2 - \phi_1}$

Complex polarization vector

Examples:

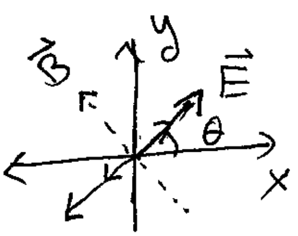
Take $\hat{k} = \hat{z}$, $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$

$$\Rightarrow \quad \vec{\epsilon} = a_1 \hat{x} + e^{i\chi} a_2 \hat{y}$$

- Linear polarization: $\boxed{\chi = m\pi}$ $(m = 0, \pm 1, \pm 2, \cdots)$

$$\Rightarrow \boxed{\vec{\epsilon} = a_1 \hat{x} \pm a_2 \hat{y}} = \text{vector in } x\text{-}y \text{ plane @ angle } \theta = \pm \tan^{-1}\left(\frac{a_2}{a_1}\right)$$

w.r.t x axis



$$\vec{E}(\vec{r}, t) = E_0 \vec{\epsilon} \cos(\quad -\omega t + \phi_1)$$

$$\vec{B}(\vec{r}, t) = \frac{E_0}{c} \hat{k} \times \vec{\epsilon} \cos(kz - \omega t + \phi_1)$$

At fixed z, $\vec{E}$ and $\vec{B}$ oscillate along a line

- Circular polarization: $\chi = \pm \frac{\pi}{2}$ $\quad a_1 = a_2 \Rightarrow a_1 = a_2 = \frac{1}{\sqrt{2}}$

$$\Rightarrow \boxed{\vec{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}(\hat{x} \pm i \hat{y})}$$

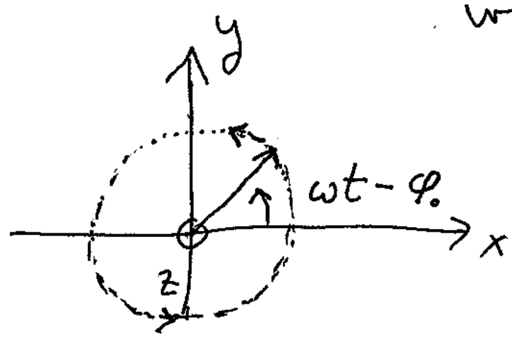
$$\vec{E}(\vec{r}, t) = \text{Re}\left(\frac{E_0}{\sqrt{2}} e^{i\phi_1} e^{i(kz - \omega t)} \hat{x} + \frac{i}{\sqrt{2}} E_0 e^{i\phi_1} e^{i(kz - \omega t)} \hat{z} \right)$$

$$\boxed{\vec{E}(\vec{r}, t) = \frac{E_0}{\sqrt{2}} \left(\cos(kz - \omega t + \phi_1) \hat{x} \mp \sin(kz - \omega t - \phi_1) \hat{y} \right)}$$

$\boxed{\text{Meaning: Consider the curve the electric field vector traces in the } x\text{-}y \text{ plane @ a fixed } z_0}$

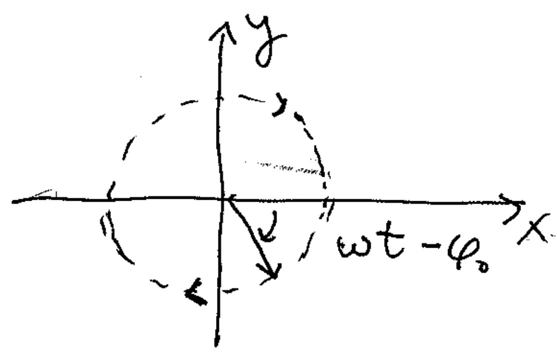
$$\Rightarrow \quad \vec{E}(z_0,t) = \frac{E_0}{\sqrt{2}}\left(\cos(\omega t - \varphi_0)\hat{x} \pm \sin(\omega t - \varphi_0)\hat{y}\right)$$

where $\varphi_0 = \phi_1 - kz_0$



Positive helicity

$$\vec{\epsilon} = \frac{\hat{x} + i\hat{y}}{\sqrt{2}}$$

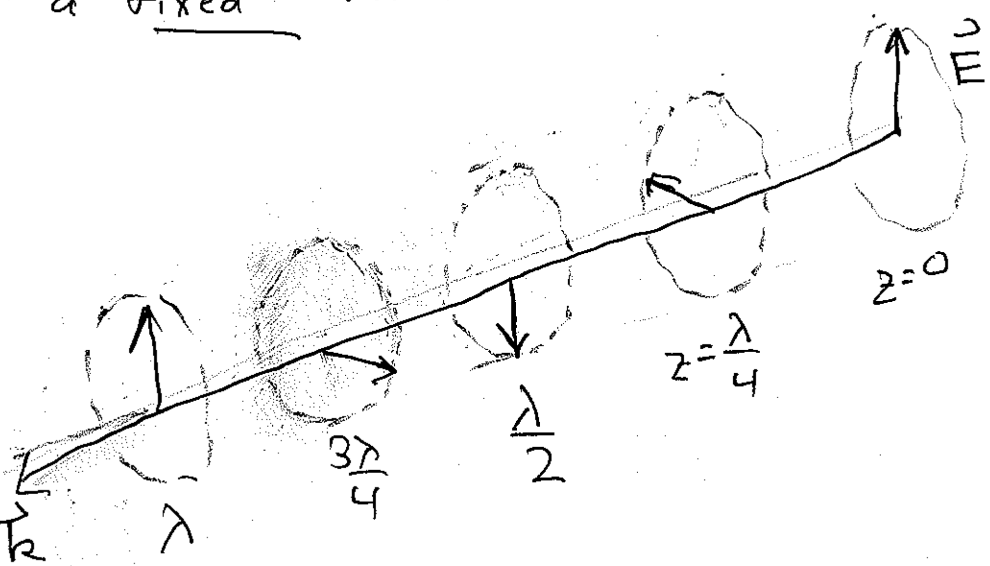


Negative helicity

$$\vec{\epsilon} = \frac{\hat{x} - i\hat{y}}{\sqrt{2}}$$

$\boxed{\text{Electric field at fixed } z_0 \text{ traces out a } \underline{\text{circle}}}$

Snap shot of electric field as a function of <u>position</u> for a <u>fixed</u> time



Positive helicity

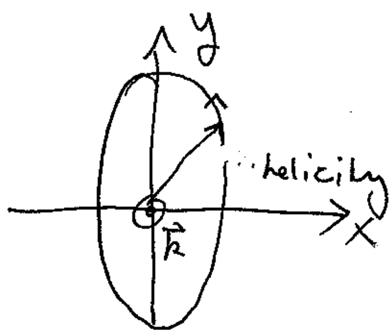
Example: $a_1 \neq a_2 \quad \chi = i\frac{\pi}{2}$, $\vec{\epsilon} = a_1 \hat{x} \pm i a_2 \hat{y}$

At fixed z_0

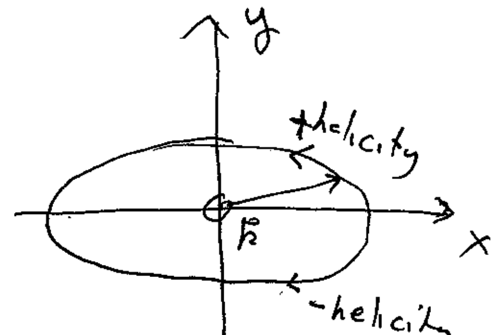
$$\Rightarrow \vec{E}(z_0,t) = E_x(t)\,\hat{e}_x + E_y(t)\,\hat{e}_y$$

$$E_x(t) = E_1 \cos(\omega t - \varphi) \qquad \pm E_y(t) = E_2 \sin(\omega t - \varphi)$$

$$\Rightarrow \left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1 \quad : \quad \underline{\text{Ellipse}}$$



$E_2 > E_1$

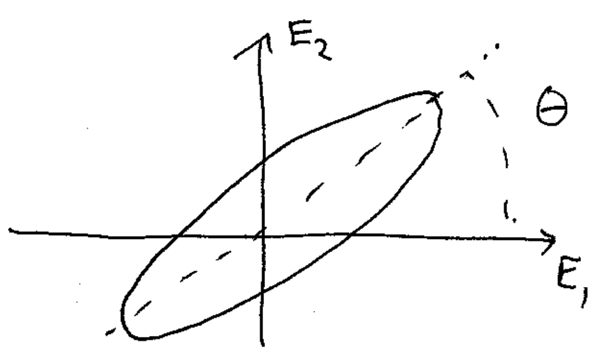


$E_1 > E_2$

Most <u>general elliptical polarization</u>

$E_1 \neq E_2 \qquad \chi$ arbitrary

$$\boxed{\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 - 2\left(\frac{E_x E_y}{E_1 E_2}\right)\cos\chi = \sin^2\chi}$$



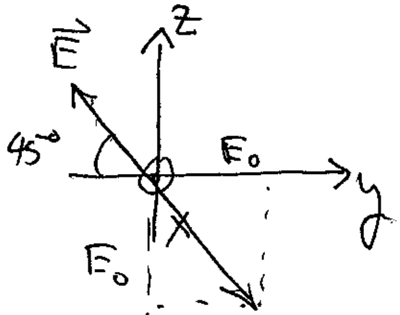
$$\Theta = \tan^{-1}\left(\frac{E_1 E_2}{E_1^2 - E_2^2}\right)$$

ellipticity : ratio of $\frac{\text{semi-minor}}{\text{semi-major}}$

Example: · Describe the polarization of the following $\vec{E}$-fields
· Sketch the curve traces by $\vec{E}$-fields
· What is the corresponding $\vec{B}$-field? Sketch $\vec{B}$

a) $\vec{E}(x,t) = E_0(-\hat{y}+\hat{z})\cos(kx-\omega t)$ (Plane wave in propagating in x-direction)

$\Rightarrow \vec{e}_1\ \vec{e}_2$ span y-z plane:



$\vec{\epsilon} = \frac{1}{\sqrt{2}}(-\hat{y}+\hat{z})$

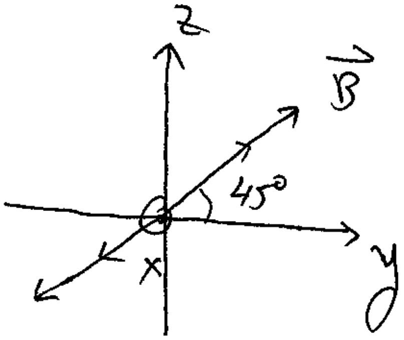
(unit polarization vector)

∴ ~~Phase~~ Real vector

$\Rightarrow$ Linear

$\vec{E}(0,t) = E_0(-\hat{y}+\hat{z})\cos\omega t$

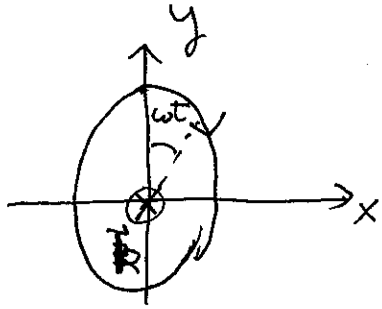
$$\vec{B} = \frac{\vec{k}}{c}\times\vec{E} = \frac{\hat{x}}{c}\times\vec{E} = \frac{E_0}{c}(-\hat{z}-\hat{y})\cos(kx-\omega t)$$



(b) $\vec{E}(z,t) = E_0\left(\hat{x}\sin(kz+\omega t) + 2\hat{y}\cos(kz+\omega t)\right)$

Plane wave propagating in the negative z-direction

Look @ $\vec{E}(z=0,t) = E_0 \left(\hat{x} \sin\omega t + 2\hat{y} \cos\omega t \right)$



Elliptical

Complex Polarization vector?

$$\vec{E}(z=0,t) = \mathrm{Re}\left(E_0 \left(i\hat{x} + 2\hat{y} \right) e^{-i\omega t} \right)$$

$$= \mathrm{Re}\left(E_0 \sqrt{5} \left(\frac{i\hat{x} + 2\hat{y}}{\sqrt{5}} \right) e^{-i\omega t} \right)$$

$$\Rightarrow \vec{\epsilon} = \frac{i\hat{x} + 2\hat{y}}{\sqrt{5}} \quad \text{(normalized)}$$

$$\vec{B}(z,t) = -\hat{z} \times \frac{\vec{E}}{c} = \frac{E_0}{c} \left(-\hat{y} \sin(kz + \omega t) + 2\hat{x} \cos(kz + \omega t) \right)$$