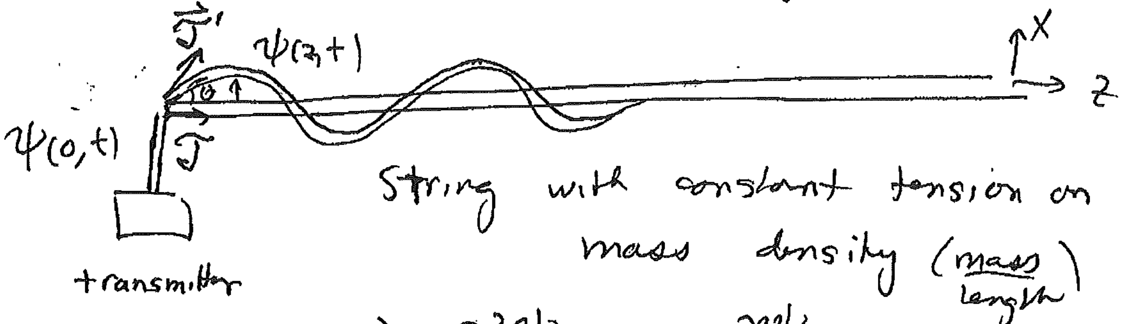
Physics 406

Lecture 10: Energy, Poynting's Theorem, Wave Propagation

Background:

Consider a wave on a string:



String with constant tension on z-axis T

mass density $\left(\frac{\text{mass}}{\text{length}}\right)$ μ_M

$$\Rightarrow \frac{\partial^2\psi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0, \qquad v = \sqrt{\frac{T}{\mu_M}}$$

from mechanics

Consider force acting on transmitter:

$$F_x(\text{string on transmitter}) = |\vec{T}'|\sin\theta = \frac{T}{\cos\theta}\sin\theta = T\tan\theta$$

$$= T\left.\frac{\partial\psi}{\partial z}\right|_{z=0}$$

For travelling wave $\psi(z,t) = \psi_0\cos(kz-\omega t)$

$$\Rightarrow \left.\frac{\partial\psi}{\partial z}\right|_{z=0} = +k\psi_0\sin(\omega t), \quad \text{But } \left.\frac{\partial\psi}{\partial t}\right|_{z=0} = -\omega\psi_0\sin\omega t$$

$$\Rightarrow F_x = -\frac{k}{\omega}T\left.\frac{\partial\psi}{\partial t}\right|_{z=0} = -\left(\frac{T}{v}\right)\left.\frac{\partial\psi}{\partial t}\right|_{z=0} = -\frac{\omega}{k}\left.\frac{\partial\psi}{\partial z}\right|_0$$

Like a <u>drag</u> force — local velocity of string

But not friction; energy goes into wave

Wave impedance $\longrightarrow$ $\boxed{\frac{T}{v} = \sqrt{T\mu_M} = Z}$

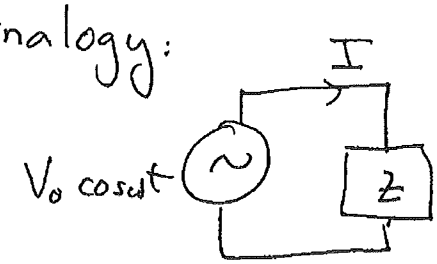
<u>Transmitted power</u>

- Damping $\Rightarrow$ String absorbs energy from transmitter
- Here, the energy is radiated down the string as a wave.
- Rate @ which transmitter does work

$$\text{Power} = F_x \left.\frac{\partial\psi}{\partial t}\right|_{z=0} = Z\left(\left.\frac{\partial\psi}{\partial t}\right|_{z=0}\right)^2$$

local velocity

Analogy:



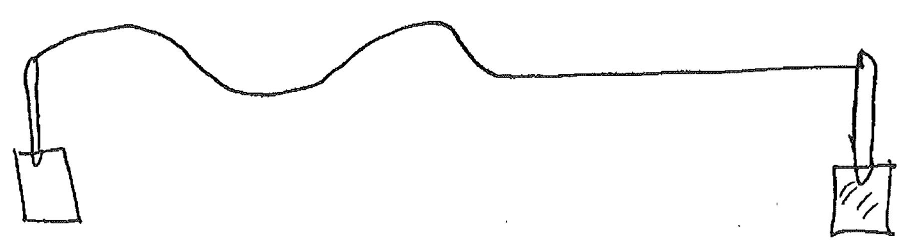
Power delivered to load impedence

$$P = VI \sim I^2 Z$$

impedence

$$V \sim F_x \qquad I \sim \frac{\partial\psi}{\partial t}$$

Energy can be recovered by a receiver



"damp pot" with some friction

Energy propagating in Electromagnetic Fields

We found that the energy stored in Electric and Magnetic fields

$$U_E = \int \underbrace{\frac{\epsilon_0}{2} |\vec{E}|^2}_{\text{"energy density" } u_E} d^3 r \quad ; \quad U_B = \int \underbrace{\frac{|\vec{B}|^2}{2\mu_0}}_{u_B} d^3 r$$

Recall from problem set # 1, the general form of a conservation law:

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = R}_{\text{"Continuity equation"}}$$

ρ = density of stuff

$\vec{J}$ = flux density : $\dfrac{\text{stuff/time}}{\text{Area}}$

$R = \dfrac{\text{rate of creating stuff}}{\text{Volume}}$

Consider total ~~e~~ electromagnetic energy density

$$u_{tot}(\vec{r},t) = u_E(\vec{r},t) + u_B(\vec{r},t)$$

$$\Rightarrow \frac{\partial}{\partial t} u_{tot} = \frac{\partial}{\partial t}\left(\epsilon_0 \frac{\vec{E}\cdot\vec{E}}{2} + \frac{1}{\mu_0}\frac{\vec{B}\cdot\vec{B}}{2}\right)$$

$$= \epsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t}$$

Substitute in from Maxwell's equations

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} \quad , \quad \frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu_0 \epsilon_0}\vec{\nabla}\times\vec{B} - \frac{1}{\epsilon_0}\vec{J}$$

$$\Rightarrow \frac{\partial}{\partial t} u_{tot} = \frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B}) + \frac{1}{\mu_0} \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{J} \cdot \vec{E}$$

$\cancel{\frac{\partial}{\partial t} u_{tot} =}$ Aside: $\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$

$$\Rightarrow \frac{\partial}{\partial t} u_{tot} = -\vec{\nabla} \cdot \left(\frac{\vec{E} \times \vec{B}}{\mu_0} \right) - \vec{J} \cdot \vec{E}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} u_{tot} + \vec{\nabla} \cdot \left(\frac{\vec{E} \times \vec{B}}{\mu_0} \right) = -\vec{J} \cdot \vec{E}}$$

Poynting's theorem!

Continuity equation for the conservation of E/M energy

$\vec{S} \equiv \frac{\vec{E} \times \vec{B}}{\mu_0}$: Poynting Vector

$$[S] = \frac{\text{Energy/time}}{\text{Area}} = \frac{\text{Power}}{\text{Area}}$$

Recall from lecture 1

$\vec{J} \cdot \vec{E}$ = rate at which field do work on charges / Volume

= "Ohmic heating"

(recall $\frac{\vec{F}}{V} = \rho \vec{E}$ $\Rightarrow$ $\frac{\vec{F} \cdot \vec{v}}{V} = \vec{J} \cdot \vec{E}$ = power dissipation

$\Rightarrow -\vec{J} \cdot \vec{E}$ = rate at which charges do work on fields

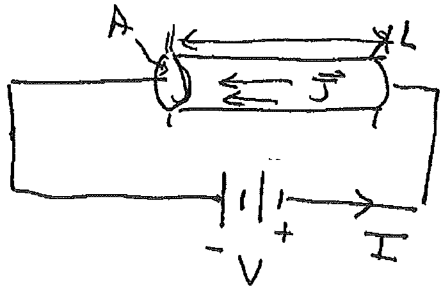
= rate of E/M energy "creation"

Integral form of Poyntings theorem

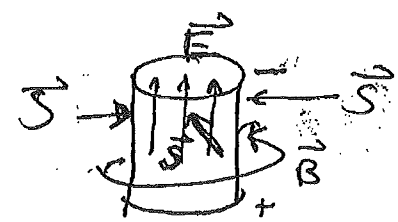
$$\frac{d}{dt}\int_V d^3r \left(\frac{\epsilon_0}{2}E^2 + \frac{B^2}{2\mu_0}\right) + \oint_S \vec{S}\cdot d\vec{a} = -\int_V d^3r\, \vec{J}\cdot\vec{E}$$

Some weirdness in the quasi-static case:

Consider Power dissipation in a resistor.



Look at resistor



$\vec{S}$ flowing into "side" of resistor

Huh?

This is a mathematical artifact.

The physical Property is $\vec{\nabla}\cdot\vec{S}$ not $\vec{S}$ alone.

so we can add any other field to $\vec{S}$ that has zero ~~curl~~ divergence and get the required physics.

In statics $\vec{\nabla}\cdot\vec{J}=0$ $\quad \frac{\partial}{\partial t}U_{total}=0$

Consider $\vec{S} \equiv \vec{J}V = \frac{I}{A}V$ (IV= power in circuit)

$$\vec{\nabla}\cdot\vec{S} = \vec{\nabla}(\vec{J}V) = \vec{J}\cdot\vec{\nabla}V = -\vec{J}\cdot\vec{E} \quad \checkmark$$

So in statics, energy flows along circuit

Energy flow in Plane Wave

Consider a travelling E-M wave in vacuum (linear polarization

$\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r}-\omega t)$; $\vec{B}(\vec{r},t) = \hat{k} \times \frac{\vec{E}_0}{c} \cos(\vec{k}\cdot\vec{r}-\omega t)$

$u_E(\vec{r},t) = \frac{\epsilon_0}{2} E_0^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$;

$u_B(\vec{r},t) = \frac{1}{2\mu_0} \frac{E_0^2}{c^2} \cos^2(\vec{k}\cdot\vec{r}-\omega t) = \frac{\epsilon_0}{2} E_0^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$

$\Rightarrow$ $\boxed{u_E(\vec{r},t) = u_B(\vec{r},t) \text{ in a plane wave}}$

Total electromagnetic energy

$u_{total}(\vec{r},t) = \epsilon_0 E_0^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$: At any $\vec{r}$ u oscillates @ ω

Typically, detection systems don't respond as fast as ω

$\Rightarrow$ We typically care about average energy over time

Time average: $\langle u_{total}(\vec{r},t)\rangle = \frac{1}{T}\int_0^T u_{total}(\vec{r},t)\, dt$

$$= \epsilon_0 E_0^2 \int_0^T \cos^2(\omega t - \vec{k}\cdot\vec{r})\, dt$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ " $\frac{1}{2}$ $\forall \vec{r}$

$$\Rightarrow \boxed{\langle u_{tot}\rangle = \frac{\epsilon_0 E_0^2}{2}}$$

Using the complex amplitude representation:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left(\underset{\sim}{\vec{E}_0}(\vec{r})\, e^{-i\omega t} \right)$$

$$\Rightarrow \langle u_E \rangle = \left\langle \frac{\epsilon_0}{2} \vec{E}\cdot\vec{E} \right\rangle \underset{\Uparrow}{=} \frac{1}{2} \mathrm{Re}\left(\frac{\epsilon_0}{2} \underset{\sim}{\vec{E}_0}^* \cdot \underset{\sim}{\vec{E}_0} \right)$$

From problem Set

$$\Rightarrow \langle u_E \rangle = \frac{\epsilon_0}{4} |\underset{\sim}{\vec{E}_0}|^2 = \frac{\epsilon_0}{4} |\vec{E}_0|^2 = \langle u_B \rangle$$

$$\Rightarrow \langle u_{total} \rangle = \frac{\epsilon_0}{2} |\vec{E}_0|^2$$

Generalize to arbitrary polarization

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left(\underset{\sim}{E_0}\, \underset{\Uparrow}{\vec{\epsilon}}\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t} \right)$$

complex polarization $\underbrace{\vec{\epsilon}^* \cdot \vec{\epsilon} = 1}_{\text{normalized}}$

$$\Rightarrow \langle u_E \rangle = \frac{\epsilon_0}{4} |\underset{\sim}{\vec{E}}(\vec{r})|^2 = \frac{\epsilon_0}{4} E_0^2$$

as before

Said another way

Real $\vec{E}(\vec{r},t) = \frac{E_0}{\sqrt{2}} \left(\cos(\vec{k}\cdot\vec{r} - \omega t)\, \hat{x} + \sin(\vec{k}\cdot\vec{r} - \omega t)\, \hat{y} \right)$

$$\Rightarrow \vec{E}\cdot\vec{E} = \frac{E_0^2}{2} \cos^2(\vec{k}\cdot\vec{r} - \omega t) + \frac{E_0}{2} \sin^2(\vec{k}\cdot\vec{r} - \omega t)$$

$$\Rightarrow \langle \vec{E}\cdot\vec{E} \rangle = \frac{E_0^2}{2}$$

: Orthogonal polarizations don't interfer

Poynting Vector of Traveling Wave

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \hat{k}\,\frac{|\vec{E}|\,|\vec{B}|}{\mu_0} = \hat{k}\,\frac{|\vec{E}_0|^2}{c\mu_0}$$

$$= \hat{k}\,\epsilon_0 c\,|\vec{E}_0|^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t) = \hat{k}\,\underbrace{c\,u_{total}(\vec{r},t)}$$

Speed × density = flux

$|\vec{S}| = c\,u_{total}$ in a plane wave

Defined Wave intensity

$$I \equiv \langle |\vec{S}| \rangle = c\,\langle u_{total} \rangle \text{ in plane wave}$$

$$\Rightarrow \boxed{I = c\left(\frac{\epsilon_0 E_0^2}{2}\right) \text{ for plane wave of electric amplitude } E_0}$$

Complex Representation:

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0}\,\mathrm{Re}(\tilde{\vec{E}} \times \tilde{\vec{B}}^*) = \frac{1}{2\mu_0 c}\,|\tilde{\vec{E}}|^2\,\hat{k}$$

Units: $[S] = \frac{\text{Watts}}{m^2}$ $\qquad [E] = \frac{\text{Volt}}{m}$

$$\left[\frac{B}{\mu_0}\right] = \frac{\text{Amp}}{m}$$

Often useful to work with H-field

recall: $\vec{H} \equiv \frac{\vec{B}}{\mu_0}$ (in vacuum)

$$\nabla \times \vec{H} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \oint \vec{H} \cdot d\vec{l} = I_{enc} \leftarrow \text{current}$$

$$\Rightarrow [H] = \frac{\text{Amp}}{\text{meter}}$$

$$\Rightarrow \boxed{\vec{S} = \vec{E} \times \vec{H}}$$

$\vec{E}$ is to V

as $\vec{H}$ is to I ← current

In free space plane wave

$$\vec{H} = \frac{\vec{B}_0}{\mu_0} \cos(\vec{k} \cdot \vec{r} - \omega t)$$

$$\vec{H} = \frac{\vec{E}}{c\mu_0} \cos(\vec{k} \cdot \vec{r} - \omega t)$$

But $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \Rightarrow \vec{H} = \sqrt{\frac{\mu_0}{\epsilon_0}}\, \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$

For plane wave in vacuum

$$\boxed{\frac{|\vec{E}_0|}{|\vec{H}_0|} = \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv 377 \text{ ohms} \equiv Z_0 \text{ ``impedance of free space''}}$$

$$\Rightarrow \text{Intensity} = \langle |\vec{S}| \rangle = \frac{1}{2} \frac{|\vec{E}_0|^2}{Z_0}$$

Example: CD player, Intensity $\mathcal{I} = 300 \frac{mW}{cm^2}$

$$\Rightarrow |\vec{E}_0|^2 = 2 Z_0 \mathcal{I} = 225 \left(\frac{\text{Volt}}{\text{cm}}\right)^2 \Rightarrow |\vec{E}_0| = 15 \frac{\text{Volt}}{\text{cm}}$$