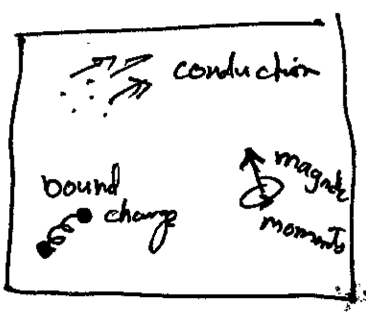
Physics 406

Lecture 12: Waves in Matter

- **Macroscopic Maxwell's Equations**

As in electro- and magneto- statics, when we are dealing with macroscopic materials, it is useful to take a coarse-grained view of sources of charge and current.



- Dielectrics
- Conductors
- Magnetic Materials

Coarse grained macroscopic sources:

$$\vec{P}(\vec{r},t) = \frac{\text{coarse-grained electric dipole moment}}{\text{Volume}} \quad \text{"polarization"}$$

$$\vec{M}(\vec{r},t) = \frac{\text{coarse-grained magnetic dipole moment}}{\text{volume}} \quad \text{"magnetization"}$$

Separate:

$$\rho_{total}(\vec{r},t) = \rho_{free}(\vec{r},t) + \rho_{bound}(\vec{r},t)$$

where $\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$

In statics, we wrote

$$\vec{J}_{total} = \vec{J}_{free} + \vec{J}_{mag}$$

where $\vec{J}_{mag} = \vec{\nabla}\times\vec{M}$ ("magnetization current")

However, there is another source of current do to time-changing bound charge. By charge conservation:

$$\frac{\partial \rho_{total}}{\partial t} = \frac{\partial \rho_{free}}{\partial t} + \frac{\partial \rho_{bound}}{\partial t} = -\vec{\nabla}\cdot\vec{J}_{total}$$

$$\stackrel{?}{=} -\underbrace{\vec{\nabla}\cdot\vec{J}_{free}}_{\frac{\partial \rho_{free}}{\partial t}} + \underbrace{\vec{\nabla}\cdot\vec{J}_{mag}}_{= \vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) = 0}$$

But $\frac{\partial \rho_{bound}}{\partial t} = -\vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t} \neq 0$ in general

$\Rightarrow$ New current source in dynamics of materials

$$\boxed{\vec{J}_{bound}(\vec{r},t) = \frac{\partial \vec{P}_{bound}}{\partial t}}$$

e.g.  oscillating bound charge driven by oscillating $\vec{E}$-field

Maxwell's Equations with "free sources" and "bound sources"

$$\vec{\nabla}\cdot\vec{E} = (\rho_{free} - \vec{\nabla}\cdot\vec{P})/\epsilon_0 \Rightarrow \vec{\nabla}\cdot(\epsilon_0\vec{E}+\vec{P}) = \rho_{free}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_{free} + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_{free} + \frac{\partial}{\partial t}(\epsilon_0\vec{E}+\vec{P})$$

<u>Define:</u> "Displacement" $\vec{D} = \epsilon_0\vec{E}+\vec{P}$

"Aux Magnetic field" $\vec{B} = \mu_0(\vec{H}+\vec{M})$

$$\Rightarrow \boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{D} = \rho_{free} & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{D}}{\partial t} \end{array}}$$

Note: Equations involving sources in terms of $\vec{D}$ and $\vec{H}$. Those without sources $\vec{E}$ and $\vec{B}$.

<u>Note:</u> $\frac{\partial\vec{D}}{\partial t} \equiv$ "Displacement current"

Linear medium: $\vec{P} = \epsilon_0 \chi_e \vec{E}$

$\vec{M} = \chi_m \vec{H}$

susceptibility

<u>Note</u>: In general $\vec{P}(\vec{r},t)$ is not instantaneously related to $\vec{E}(\vec{r},t)$. There is a "response time". More general expression:

$$\vec{P}(\vec{r},t) = \epsilon_0 \int dt' \, \chi_e(t-t') \, \vec{E}(\vec{r},t')$$

"response function"

Most often, $\vec{P}$ is the "steady-state" response to an oscillating field @ freq ω. For that frequency

$$\underset{\sim}{\vec{P}}(\vec{r},\omega) = \epsilon_0 \, \underset{\sim}{\chi_e}(\omega) \, \underset{\sim}{\vec{E}}(\vec{r},\omega)$$

Complex amplitude of polarization

complex response function

complex amplitude

$$\vec{P}(\vec{r},t) = \mathrm{Re}\left(\underset{\sim}{\vec{P}}(\vec{r},\omega)\, e^{-i\omega t}\right)$$

Formally, $\underset{\sim}{\chi_e}(\omega)$ is the Fourier transform of $\chi_e(\tau)$. More on this later.

Ignoring response time, or thinking of a single frequency, we define

$$\underset{\substack{\uparrow \\ \text{dielectric permittivity}}}{\epsilon} = \epsilon_0 (1+\chi_e) = \epsilon_0 \underset{\substack{\nwarrow \\ \text{dieletric constant}}}{K_e}$$

$$\mu \equiv \mu_0 (1+\chi_m) = \mu_0 K_m$$

$$\Rightarrow \quad \vec{D} = \epsilon \vec{E} \qquad \vec{B} = \mu \vec{H}$$

$\Rightarrow$ Maxwells eqns in linear media (fast response)

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho_f/\epsilon & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \mu \vec{J}_f + \mu\epsilon \dfrac{\partial \vec{E}}{\partial t} \end{array}}$$

<u>Energy in Linear Media</u> $\epsilon_0 \to \epsilon$, $\mu_0 \to \mu$

$$\left.\begin{array}{l} u_E = \dfrac{\epsilon}{2}\vec{E}\cdot\vec{E} = \dfrac{\vec{D}\cdot\vec{E}}{2} \\ u_B = \dfrac{1}{2\mu}\vec{B}\cdot\vec{B} = \dfrac{\vec{B}\cdot\vec{H}}{2} \end{array}\right\} \text{Energy density}$$

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu} = \vec{E}\times\vec{H} : \text{Poynting vector}$$

$$\frac{\partial}{\partial t}(u_E + u_B) + \vec{\nabla}\cdot\vec{S} = -\vec{J}_{free}\cdot\vec{E}$$

Waves in linear media in the absense of free charge

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

• Deriving the wave equation

$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{E})}_{\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}}) - \nabla^2\vec{E}} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$\vec{\nabla}\cdot\vec{E} \to 0$

$$\Rightarrow \boxed{\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0}$$

Similarly

$$\boxed{\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0}$$

Wave equation with phase velocity $\boxed{v = \frac{1}{\sqrt{\mu\epsilon}}}$

$$\Rightarrow v = \frac{1}{\sqrt{\mu_0\epsilon_0 K_m K_e}} = \frac{c}{\sqrt{K_m K_e}}$$

Define index of refraction: $\boxed{n \equiv \frac{c}{v} = \sqrt{K_e K_m}}$

In most materials, non-magnetic $\Rightarrow K_m \approx 1$

Typically $\boxed{n = \sqrt{K_e}}$

<u>Plane-wave modes:</u>
$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\vec{B} = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Plug into wave equation to get "dispersion relation" $\omega(k)$

$$\Rightarrow \quad -k^2 + \mu\epsilon\,\omega^2 = 0 \quad \Rightarrow \quad k = \sqrt{\mu\epsilon}\,\omega = \frac{\omega}{v}$$

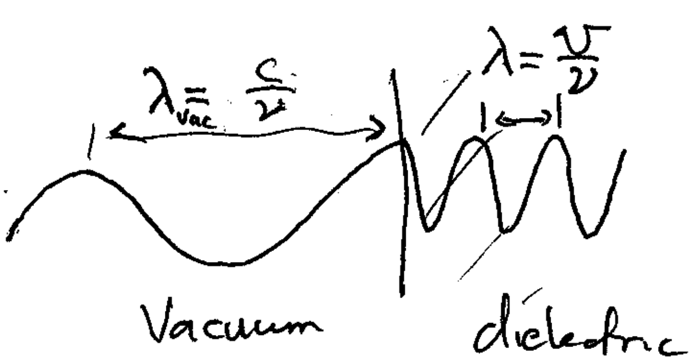
$$\boxed{k = \frac{\omega}{v} = n\frac{\omega}{c}}$$

Note: generally, $n(\omega)$: Index of refraction depends on ω (more soon)

Inside medium: $\lambda = \dfrac{2\pi}{k} = \dfrac{c}{n}\left(\dfrac{2\pi}{\omega}\right) = \dfrac{1}{n}\underbrace{\left(\dfrac{c}{\nu}\right)}_{\lambda_{vac}}$

$$\Rightarrow \quad \boxed{\lambda = \frac{\lambda_{vac}}{n}}$$ Wavelength inside medium modified compared to what it is in vacuum.

<u>Note</u>: Frequency is unmodified from its vacuum value; determined by <u>source</u>



Other relations for plane waves in dielectrics:

$$\vec{\nabla}\cdot\vec{E} = 0 \implies \vec{k}\cdot\vec{E}_0 = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0 \implies \vec{k}\cdot\vec{B}_0 = 0$$

Wave is "transverse" inside dielectric

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \implies i\vec{k}\times\vec{E}_0 = i\omega\vec{B}_0$$

$$\implies \vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0 = \hat{k}\times\frac{\vec{E}_0}{v}$$

$$\implies \boxed{|\vec{B}_0| = \frac{|\vec{E}_0|}{v} = n\frac{|\vec{E}_0|}{c}}$$

$$|\vec{H}_0| = \frac{|\vec{E}_0|}{\mu v} = \sqrt{\frac{\epsilon}{\mu}}|\vec{E}_0|$$

Energy density:

$$u_E = \frac{\epsilon}{2}\vec{E}\cdot\vec{E} \implies \langle u_E\rangle = \frac{\epsilon}{4}\mathrm{Re}(\vec{E}\cdot\vec{E}^*) = \frac{\epsilon}{4}|\vec{E}_0|^2$$

$$u_B = \frac{1}{2\mu}\vec{B}\cdot\vec{B} \implies \langle u_B\rangle = \frac{1}{4\mu}\mathrm{Re}(\vec{B}\cdot\vec{B}^*) = \frac{1}{4\mu}|\vec{B}_0|^2$$

$$= \frac{1}{4\mu v^2}|\vec{E}_0|^2 = \frac{\epsilon}{4}|\vec{E}_0|$$

$$\implies \boxed{\langle u_E\rangle = \langle u_B\rangle}$$ in linear medium with ϵ, μ real

Poynting vector: $\vec{S} = \vec{E}\times\vec{H} = \frac{\vec{E}\times\vec{B}}{\mu}$

$\implies$ Intensity $I = \frac{1}{2}\mathrm{Re}\left(\frac{\vec{E}\times\vec{B}^*}{\mu}\right) = \frac{1}{2}\frac{E_0 B_0}{\mu}\hat{k} = v\frac{\epsilon E_0^2}{2}\hat{k}$

$$\implies \boxed{I = v\, U_{total}\,\hat{k}}$$

$$\boxed{H_0 = \frac{E_0}{Z} \qquad Z = \sqrt{\frac{\mu}{\epsilon}}}$$