Physics 406

Lecture 13: Reflection and Refraction

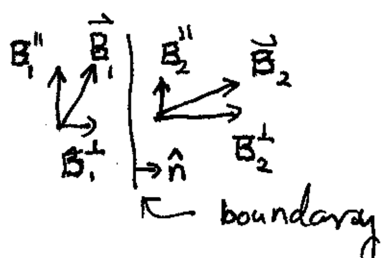
Boundary Conditions:

As we discussed in the context of statics, the field equations determine boundary conditions on the components of $\vec{E}$ and $\vec{B}$, $\vec{D}$ and $\vec{H}$, at any surface:

Macroscopic Maxwell:

$$\boxed{\begin{array}{c|c} \vec{\nabla}\cdot\vec{D} = \rho_{free} & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{D}}{\partial t} \end{array}}$$

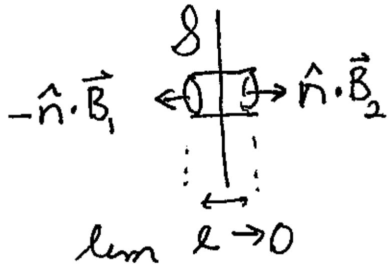
We break up the fields in a component normal (perpendicular) to the surface and a component tangential (parallel) to surface

e.g. 

We use the integral form Maxwells equations to pick out components:

Normal component: Use pill box shrunk ~~so~~ to straddle surface so no flux through side

Tangential Component: Use contour shrunk to straddle surface so no circulation normal part



$$\oint_{\mathcal{S}} \vec{B}\cdot d\vec{a} = 0 \iff \vec{\nabla}\cdot\vec{B} = 0$$

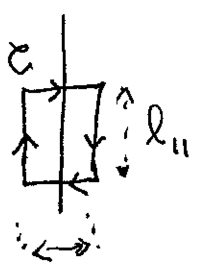
$\underset{\ell \to 0}{\text{limit}}$, $\underset{\text{Cap Area}}{\lim} \to 0$

$$\Rightarrow A\left(B_2^{\perp} - B_1^{\perp}\right) = 0 \Rightarrow \boxed{B_1^{\perp} = B_2^{\perp}}$$

$$\oint_{\mathcal{S}} \vec{D}\cdot da = Q_{\text{free}}^{\text{enclosed}} \iff \vec{\nabla}\cdot\vec{D} = \rho_{\text{free}}$$

$$\Rightarrow A\left(D_2^{\perp} - D_1^{\perp}\right) = Q_{\text{free}}^{\text{enclosed}} \Rightarrow \boxed{D_2^{\perp} - D_1^{\perp} = \sigma_{\text{free}}}$$

surface charge at boundary



limit of $\ell_{\perp} \to 0$

$$\oint_{C} \vec{E}\cdot d\ell = -\frac{d}{dt}\Phi_B \iff \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$\underset{\ell_{\perp}\to 0,\ \ell_{\parallel}\to 0}{\lim}$ $\Phi_B \to 0$

$$\oint_{C} \vec{E}\cdot d\vec{\ell} = \ell_{\parallel}\left(E_1^{\parallel} - E_2^{\parallel}\right) = 0 \Rightarrow \boxed{E_1^{\parallel} = E_2^{\parallel}}$$

$$\oint_{C} \vec{H}\cdot d\vec{\ell} = I_{\text{free}}^{\text{enc}} + \cancel{\frac{d}{dt}\int \vec{D}\cdot d\vec{a}}^{\,0}$$ in limit of shrinking contours

$$\Rightarrow \ell_{\parallel}\left(H_1^{\parallel} - H_2^{\parallel}\right) = I_{\text{free}}^{\text{enc}} \Rightarrow \boxed{H_1^{\parallel} - H_2^{\parallel} = K_{\text{free}}}$$

surface current density

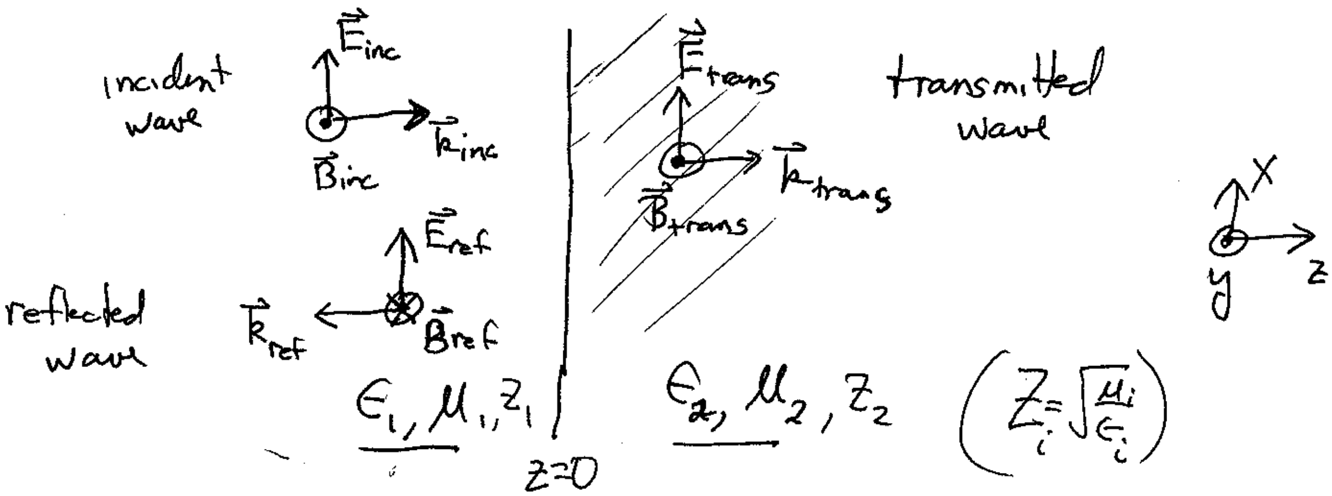
In linear media with no free charge or current

$\vec{B} = \mu \vec{H}$ $\quad \vec{D} = \epsilon \vec{E}$ $\quad \sigma_{free} = K_{free} = 0$

$\Rightarrow$

$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$	$B_1^{\perp} = B_2^{\perp}$
$E_1^{\parallel} = E_2^{\parallel}$	$\dfrac{B_1^{\parallel}}{\mu_1} = \dfrac{B_2^{\parallel}}{\mu_2}$

Reflection of plane waves at the interface between linear media (dielectrics)



$\left(Z_i = \sqrt{\frac{\mu_i}{\epsilon_i}} \right)$

Incident Wave

$\vec{E}_{inc}(z,t) = E_{0_i} e^{i(k_i z - \omega t)} \hat{x}$

$\vec{H}_{inc}(z,t) = \dfrac{E_{0i}}{Z_1} e^{i(k_i z - \omega t)} \hat{y}$

Similar for transmitted

Reflected wave

$\vec{E}_{ref}(z,t) = E_{0_r} e^{i(-k_r z - \omega t)} \hat{x}$

$\vec{H}_{ref}(z,t) = \dfrac{E_{0r}}{Z_1} e^{i(-k_r z - \omega t)} \hat{y}$

For normal incidence, at boundary only ∥-components

$$\vec{E}_1(z=0,t) = \vec{E}_{inc}(z=0,t) + \vec{E}_{ref}(z=0,t)$$

$$\vec{E}_2(z=0,t) = \vec{E}_{trans}(z=0,t)$$

Similar for $\vec{H}$

All wave have same time dependence

<u>Boundary conditions</u>

$$E_1^{\parallel} = E_2^{\parallel} \implies E_{0i} + E_{0r} = E_{0t}$$

$$H_1^{\parallel} = H_2^{\parallel} \implies \frac{E_{0i}}{Z_1} - \frac{E_{0r}}{Z_1} = \frac{E_{0t}}{Z_2}$$

Define: $\begin{cases} t \equiv \dfrac{E_{0t}}{E_{0i}} & \text{(transmission amplitude)} \\ r \equiv \dfrac{E_{0r}}{E_{0i}} & \text{(reflection amplitude)} \end{cases}$

$$\implies 1 + r = t \qquad \frac{1}{Z_1} - \frac{r}{Z_1} = \frac{t}{Z_2}$$

Solve

$$\implies \frac{1-r}{Z_1} = \frac{1+r}{Z_2} \implies \boxed{r = \frac{Z_2 - Z_1}{Z_1 + Z_2}} \qquad Z_i = \sqrt{\frac{\mu_i}{\epsilon_i}}$$

non-perfect transmission due to impedance mismatch

$$\boxed{t = \frac{2Z_2}{Z_1 + Z_2}}$$

In many ~~solids~~ (most) materials $\mu = \mu_0$ (non-magnetic)

$$\Rightarrow \quad Z = \sqrt{\frac{\mu_0}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{\sqrt{K_e}} = \frac{Z_0}{n} \qquad \left(\begin{array}{c} n \approx \sqrt{K_e} \\ \text{index of refraction} \end{array} \right)$$

$$\Rightarrow r = \frac{\frac{Z_0}{n_2} - \frac{Z_0}{n_1}}{\frac{Z_0}{n_1} + \frac{Z_0}{n_2}} \Rightarrow \boxed{r = \frac{n_1 - n_2}{n_1 + n_2}}$$

$$t = \frac{2\frac{Z_0}{n_2}}{\frac{Z_0}{n_1} + \frac{Z_0}{n_2}} \Rightarrow \boxed{t = \frac{2n_1}{n_1 + n_2}}$$

reflection/transmission for non-magnetic materials (index mismatch)

Often, what we are interested in is the fraction of intensity that reflected or transmitted

since ~~[illegible]~~ $I = \frac{1}{2} \frac{|\vec{E}|^2}{Z}$

Define $R = \frac{I_{ref}}{I_{inc}}$ $\quad T = \frac{I_{trans}}{I_{inc}}$

$$\Rightarrow \boxed{R = \frac{|E_{or}|^2}{|E_{oi}|^2} = |r|^2 = \frac{(Z_2 - Z_1)^2}{(Z_1 + Z_2)^2} \underset{\mu = \mu_0}{=} \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2}}$$

$$\boxed{T = \frac{\frac{|E_{ot}|^2}{Z_2}}{\frac{|E_{oi}|^2}{Z_1}} = \frac{Z_1}{Z_2}|t|^2 = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} \underset{\mu = \mu_0}{=} \frac{4n_1 n_2}{(n_1 + n_2)^2}}$$

By energy conservation, we must have

$$I_{inc} = I_{ref} + I_{trans}$$

$$\Rightarrow \quad R + T = 1$$

<u>Check</u>

$$\frac{(Z_2 - Z_1)^2}{(Z_1 + Z_2)^2} + \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2} = \frac{(Z_1 + Z_2)^2}{(Z_1 + Z_2)^2} = 1 \quad \checkmark$$

Example: Glass $\mu \approx 1 \qquad n = 1.5$

$$R = \left(\frac{1 - 1.5}{1 + 1.5}\right)^2 = 0.04$$

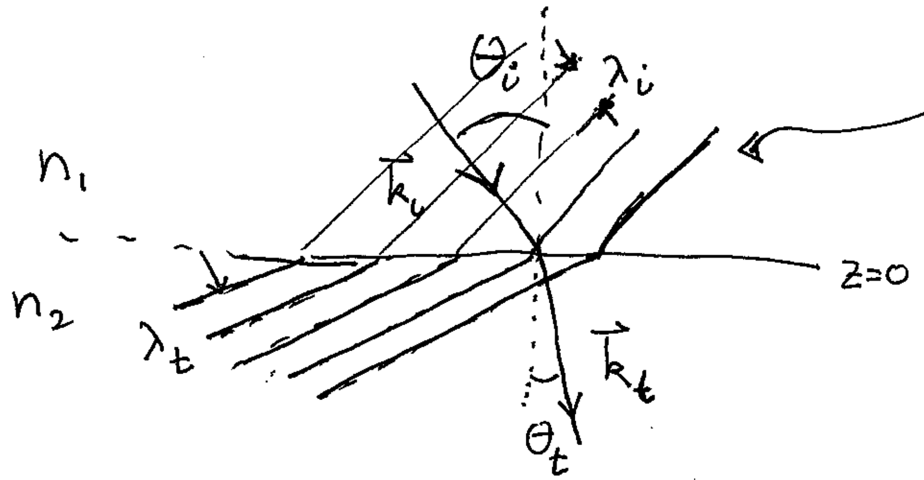
, only 4% of intensity is reflected

Note: For non-magnetic materials $\mu \neq \mu_0$

Can have $n_1 = n_2$ but $Z_1 \neq Z_2$.

Reflection due to impedance mismatch, not index mismatch (though they are the same when $\mu \approx \mu_0$)

Oblique Incidence (Refraction)



Wave front must be continuous (~~point~~ surface of constant phase of signal)

Sketch here where $\lambda_t < \lambda_i \Rightarrow$ Index $n_2 > n_1$

- Incident wave $\vec{E}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$
- Reflected wave $\vec{E}_{0r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$
- Transmitted wave $\vec{E}_{0t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$

Continuity of phase at boundary, $z=0$ (flat plane)

$$\Rightarrow \vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} = \vec{k}_t \cdot \vec{r} \Big|_{z=0}$$

$$\Rightarrow |\vec{k}_i| \sin\theta_i = |\vec{k}_r| \sin\theta_r = |\vec{k}_t| \sin\theta_t$$

$$|\vec{k}_i| = |\vec{k}_r| = \frac{\omega}{c} n_1 \quad \Rightarrow \boxed{\sin\theta_i = \sin\theta_r} \Rightarrow \boxed{\theta_i = \theta_r}$$

$\Rightarrow$ At flat ~~Plane~~ Plane, angle of incidence equals angle of reflection

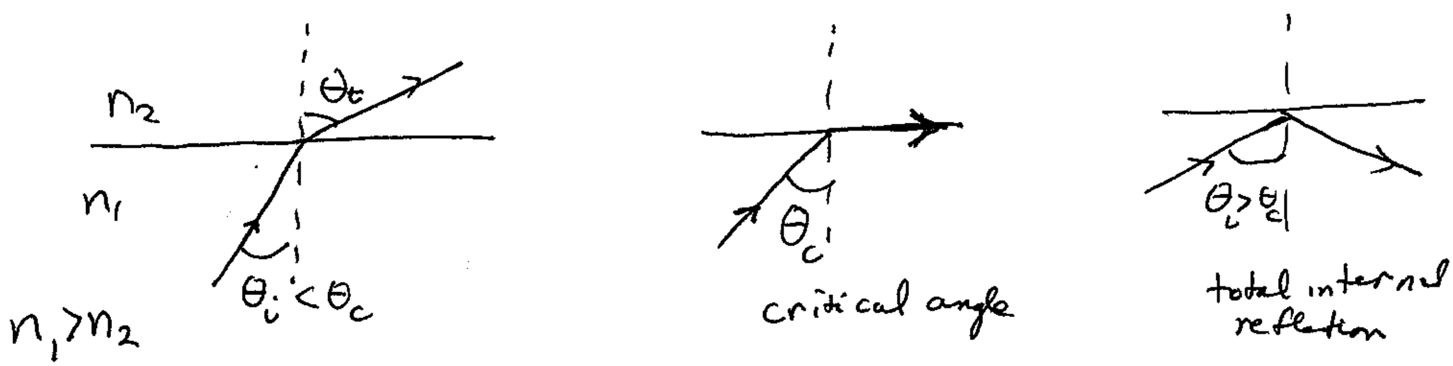
$$|\vec{k}_t| = \frac{\omega}{c} n_2$$

$$\Rightarrow \boxed{n_1 \sin\theta_i = n_2 \sin\theta_t} \quad \text{Snell's law}$$

law of refraction

these laws are true for all waves, not just EM

Total internal reflection



where $\theta_i = \theta_c$, $\theta_t = \frac{\pi}{2}$ $\Rightarrow$ $\boxed{\sin\theta_c = \frac{n_2}{n_1}}$ Critical angle

For glass and air $n_1 \approx 1$, $n_2 \approx 1.5$ $\Rightarrow$ $\theta_c \approx 41.8°$

What happens to transmitted wave for $\theta_i > \theta_c$?

By Snells law $\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{\sin\theta_i}{\sin\theta_c} > 1$??

$\Rightarrow \theta_t$ not a real number (next page)

$$\cos\theta_t = \pm\sqrt{1-\sin^2\theta_t} = \pm\sqrt{1-\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2}$$

$$= \pm i \underbrace{\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}}_{\text{real}}$$

Transmitted wave

$$\vec{E}_{trans} = \vec{E}_{0t}\, e^{i(\vec{k}_t\cdot\vec{r} - \omega t)} = \vec{E}_{0t}\, e^{i(k_t z\cos\theta_t + k_t x\sin\theta_t - \omega t)}$$

$$= \vec{E}_{0t}\, \underbrace{e^{-k_t z\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}}}_{\text{Exponential decay!}}\; e^{i(k_t x\sin\theta_t - \omega t)}$$

No propagation along z-direction

"Evanescent wave" (like classically forbidden region in quantum)

Intensity very close to surface (surface wave)

$$I = I_{0t}\, e^{-2k_t\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2-1}\; z}$$

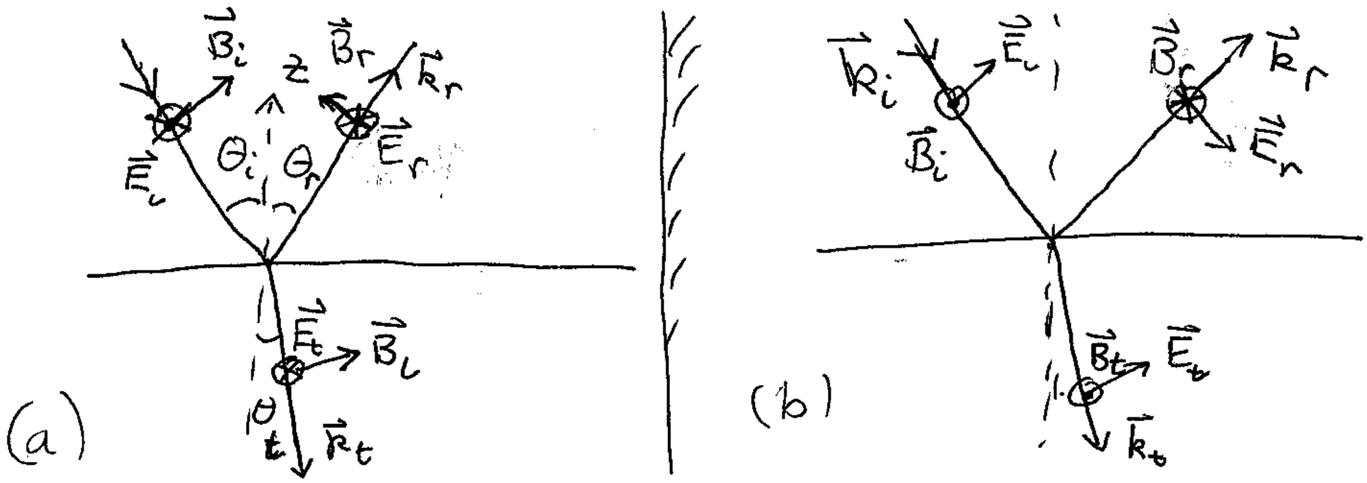
"Propagation distance" $d = \dfrac{1}{2k_t\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2-1}}$

$$\boxed{d = \frac{\lambda_{vac}}{n_2\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}}}$$

Reflection / Transmission Coefficients at oblique Incidence

Two cases to consider:

(a) Electric field polarized $\perp$ to "plane of incidence"
(b) Electric field polarized $\parallel$ to "plane of incidence"



Boundary conditions

(a) $E_1^{\parallel} = E_2^{\parallel} \Longrightarrow E_{0i} + E_{0r} = E_{0t}$

$$\Rightarrow 1 + r = t$$

$$H_1^{\parallel} = H_2^{\parallel} \Rightarrow \frac{E_{0i}}{Z_1}\cos\theta_i - \frac{E_{0r}}{Z_1}\cos\theta_i = \frac{E_{0t}}{Z_2}\cos\theta_t$$

$$\Rightarrow \cos\theta_i(1-r) = \frac{Z_1}{Z_2} t \cos\theta_t$$

Solve $\Rightarrow$ $$\boxed{r_a = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}}$$

Non magnetic material

$$\boxed{r_a = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}}$$

where

$$\boxed{\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i}$$

<u>case (b)</u>

$$E_1^{\parallel} = E_2^{\parallel} \Longrightarrow E_{oi}\cos\theta_i + E_{or}\cos\theta_i = E_{ot}\cos\theta_t$$

$$H_1^{\parallel} = H_2^{\parallel} \Longrightarrow \frac{E_{oi}}{Z_1} - \frac{E_{or}}{Z_1} = \frac{E_{ot}}{Z_2}$$

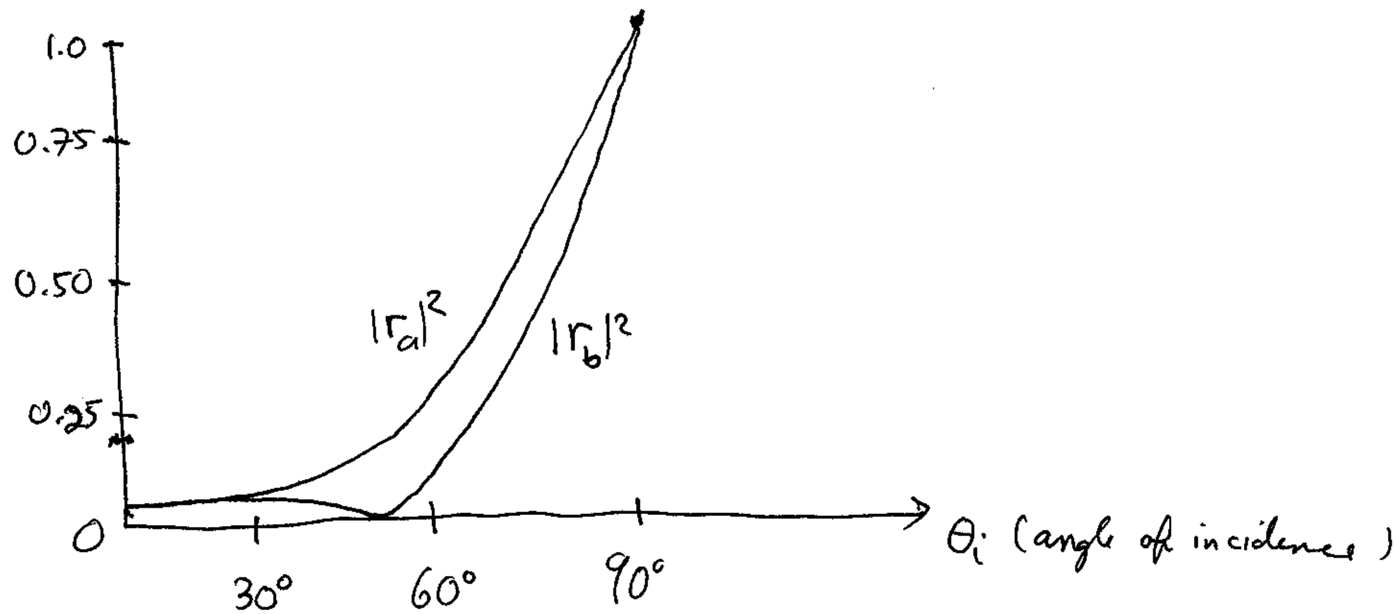
$$\Rightarrow \begin{cases} \cos\theta_i\,(1+r) = t\cos\theta_t \\ \frac{1}{Z_1}(1-r) = \frac{1}{Z_2}\,t \end{cases}$$

$$\Rightarrow Z_1\left(\frac{1+r}{1-r}\right)\cos\theta_i = Z_2\cos\theta_t$$

$$\Rightarrow \boxed{r_b = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i} \overset{\Uparrow}{=} \frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}}$$

non-magnetic

Sketch of graph with $n_1 = 1$ $n_2 = 1.5$ (glass)



We see that the reflection coefficients are different for the two different polarizations.

For electric field polarized in the plane of incidence, at ~~and~~ an angle of incidence s.t. $\tan\theta_i = \frac{n_2}{n_1}$ the reflection coefficient is <u>zero</u>. This is known as <u>Brewster's angle</u> and is a consequence of the fact that E-M waves are transverse (more later).