Physics 406

Lecture 14: Electromagnetic Waves in Conductors

In a conductor there are free as well as bound charges.

We will consider an "ohmic" medium so that the free current is determined by "Ohm's Laws"

$$\vec{J}_f = \sigma \vec{E} \quad \text{(conductivity)}$$

Note: Like the dielectric susceptibility there is a finite "response time". Thus this proportionality only makes sense in the steady state of a sinusoidal drive (frequency domain), or for very fast response.

We this caveat, and in a linear medium with μ, ϵ Maxwell's eqns. read:

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = \rho_f/\epsilon \qquad && \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &\vec{\nabla}\cdot\vec{B} = 0 && \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t} \end{aligned}}$$

Maxwell's Eqns in a conductor

Now, from the continuity Eqn.

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{J}_f = -\sigma\,\vec{\nabla}\cdot\vec{E} \quad \text{in conductor bulk}$$

$$\Rightarrow \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\epsilon}\rho_f \Rightarrow \rho_f(\vec{r},t) = e^{-\frac{\sigma}{\epsilon}t}\rho_f(\vec{r},0)$$

$\Rightarrow$ Any initial free bulk charge density in the conductor decays to zero with a characteristic time (ϵ/σ). Any residual free charge density must reside at the surface.

Good conductor ϵ/σ very small $\Rightarrow$ Very short transient.

$\Rightarrow$ Set $\rho_f \to 0$ but not $J_f = \sigma\vec{E}$

$$\Rightarrow \boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = 0 & \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} + \mu\sigma\vec{E} \end{array}}$$

Maxwell's Eqns in conductor after time $t \gg \epsilon/\sigma$

Deriving Wave Eqn:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}})^{0} - \nabla^2\vec{E} = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) = -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t}$$

$$\boxed{\begin{aligned} &\Rightarrow \quad \nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \\ &\text{Similarly} \quad \nabla^2\vec{B} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{B}}{\partial t} \end{aligned}}$$

New terms, w/ first time derivative $\propto$ conductivity

Look for plane wave solutions

$$\underset{\sim}{\vec{E}}(\vec{r},t) = \underset{\sim}{\vec{E}_0}\, e^{i(\underset{\sim}{\vec{k}}\cdot\vec{r}-\omega t)} \qquad \underset{\sim}{\vec{B}}(\vec{r},t) = \underset{\sim}{\vec{B}_0}\, e^{i(\underset{\sim}{\vec{k}}\cdot\vec{r}-\omega t)}$$

Allow complex

$$\Rightarrow -\tilde{k}^2 = -\mu\epsilon\omega^2 - i\mu\sigma\omega \quad \Rightarrow \quad \text{Wave number is } \underline{\text{complex}}$$

$$\tilde{k} = \sqrt{\mu\epsilon\omega^2 + i\mu\sigma\omega}, \quad \text{where} \quad \tilde{n} = \sqrt{K_e K_m}\sqrt{1 + i\frac{\sigma}{\epsilon\omega}}$$

$$= \frac{\omega}{c}\tilde{n}(\omega)$$

<u>Complex index of refraction</u>

having used $\frac{1}{\sqrt{\mu_0\epsilon_0}} = c$

Meaning of complex wave number

$$\tilde{k} \equiv \underset{\substack{\uparrow \\ \text{real part}}}{k_R} + i \underset{\substack{\uparrow \\ \text{imaginary part}}}{k_I}$$

Let propagation direction be z-direction, and $\vec{E}_0 = E_0 \hat{x}$

$$e^{i\tilde{k}z} = e^{ik_R z - k_I z} = e^{-k_I z} e^{ik_R z}$$

$$\Rightarrow \boxed{\vec{E}(z,t) = E_0 e^{-k_I z} e^{i(k_R z - \omega t)} \hat{x}}$$

Real part: Propagation of wave
Imaginary part: Attenuation of wave (or "gain" if < 0)

$$k_R = \frac{\omega}{c} n_R(\omega) \qquad k_I = \frac{\omega}{c} n_I(\omega)$$

where $\hat{n}(\omega) = n_R(\omega) + i n_I(\omega)$

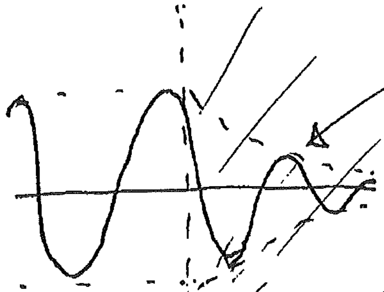
Here $\tilde{n}(\omega) = \underbrace{\sqrt{K_m K_e}}_{\equiv n_0} \left(1 + i\frac{\sigma}{\epsilon\omega}\right)^{1/2}$

Generall expression:

$$n_R(\omega) = n_0 \sqrt[4]{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \cos \phi/2$$

$$n_I(\omega) = n_0 \sqrt[4]{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \sin \phi/2$$

where $\phi = \tan^{-1}\left(\frac{\sigma}{\epsilon\omega}\right)$



Vacuum | conductor

decaying envelope of amplitude $e^{-\kappa z}$

~~Peak~~ Penetration distance

$\equiv$ "skin depth"

$$\delta \equiv \frac{1}{k_I} = \frac{c}{\omega}\frac{1}{n_I(\omega)}$$

$$\boxed{\delta = \frac{\lambda_{vac}}{2\pi}\frac{1}{n_I(\omega)}}$$

In a "good conductor" $\frac{\sigma}{\epsilon} \gg \omega$ at reasonable frequencies

$$\Rightarrow \tilde{n}(\omega) \approx n_0\left(\frac{i\sigma}{\epsilon\omega}\right)^{1/2} = n_0\sqrt{\frac{\sigma}{\epsilon\omega}}\,(i)^{1/2}$$

Aside: $i^{1/2} = \left(e^{i\frac{\pi}{2}}\right)^{1/2} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$

$$\Rightarrow n_R(\omega) \approx n_I(\omega) = \sqrt{\frac{\sigma}{2\epsilon_0\omega}} \gg 1 \quad \text{(having set } K_m \approx 1\text{)}$$

Then $k_R(\omega) \approx k_I(\omega) \Rightarrow$ Propagation distance not even a wavelength (skin)

$\Rightarrow$ Good conductor $\delta = \frac{c}{\omega}\sqrt{\frac{2\epsilon_0\omega}{\sigma}}$

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}}$$

Example: Silver $\sigma = 6.14 \times 10^7\ (\text{ohm m})^{-1}$

$\Rightarrow$ at optical frequency $\omega = 10^{15}\ s^{-1}$ $\boxed{\delta \cong 5\ \text{nm}}$

Nature of waves in conductors

- **Transverse**, since $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$

- Relation between $\tilde{E}_0$ and $\tilde{B}_0$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow i\tilde{\vec{k}}\times\tilde{\vec{E}}_0 = i\omega\tilde{\vec{B}}_0$$

$$\Rightarrow \quad \tilde{\vec{B}}_0 = \frac{\tilde{\vec{k}}\times\tilde{\vec{E}}_0}{\omega} \qquad \tilde{\vec{k}} \perp \vec{E}_0$$

$$\Rightarrow \quad \tilde{B}_0 = \frac{\tilde{k}}{\omega}\tilde{E}_0 = \frac{|\tilde{k}|}{\omega}e^{i\phi}\tilde{E}_0 \qquad \text{where } \phi = \tan^{-1}\left(\frac{k_I}{k_R}\right)$$

$\Rightarrow$ $\tilde{\vec{B}}_0$ and $\tilde{\vec{E}}_0$ <u>not in phase</u> in absorbing medium

$$\vec{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t)\,\hat{x}$$

$$\vec{B}(z,t) = B_0 e^{-\kappa z}\cos(kz-\omega t+\phi)\,\hat{y}$$

this phase difference has implications for the propagation of energy

Reflection from a conductor

We saw in Lecture 13, for normal incidence, the reflection amplitude is

$$r = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad \text{where } Z \equiv \frac{E_0}{H_0} \text{ is the wave impedance in the medium}$$

What is Z in a conductor?

We must now deal with complex amplitudes

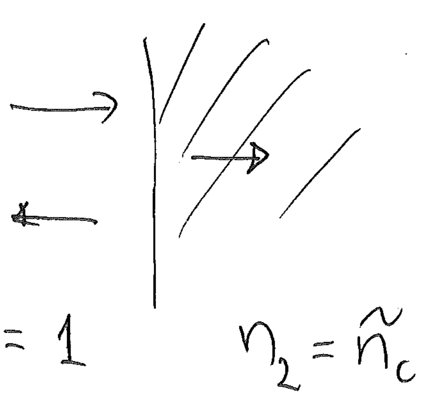
$$\tilde{Z}_c = \frac{\tilde{E}_0}{\tilde{H}_0} = \mu \frac{\tilde{E}_0}{\tilde{B}_0} = \mu \frac{\omega}{\tilde{k}} = \frac{\mu c}{\tilde{n}} = \frac{\mu}{\sqrt{\mu_0 \epsilon_0}} \frac{1}{\tilde{n}}$$

Let us consider non-magnetic materials $\mu \approx \mu_0$

$$\Rightarrow \quad \tilde{Z}_c \cong \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{\tilde{n}} = \frac{Z_0}{\tilde{n}}$$

$$\tilde{n}_c = n_0 \left(1 + i\frac{\sigma}{\omega\epsilon}\right)^{1/2}, \quad n_0 = \sqrt{K_e}$$

Reflection from conductor, incident from vacuum



$$\tilde{r} = \frac{1 - \tilde{n}_c}{1 + \tilde{n}_c}$$

For a good conductor, $\left(\frac{\sigma}{\epsilon_0} \gg \omega\right)$, $\tilde{n}_c \approx \sqrt{\frac{\sigma}{\epsilon_0 \omega}}\, e^{i\pi/4}$

$$\Rightarrow \quad \tilde{r} \approx \frac{1 - \sqrt{\frac{\sigma}{\epsilon_0 \omega}}\, e^{i\pi/4}}{1 + \sqrt{\frac{\sigma}{\epsilon_0 \omega}}\, e^{i\pi/4}} = \frac{-1 + \sqrt{\frac{\epsilon_0 \omega}{\sigma}}\, e^{-i\pi/4}}{1 + \sqrt{\frac{\epsilon_0 \omega}{\sigma}}\, e^{-i\pi/4}}$$

$$\approx \left(-1 + \sqrt{\frac{\epsilon_0 \omega}{\sigma}}\, e^{-i\pi/4}\right)\left(1 - \sqrt{\frac{\epsilon_0 \omega}{\sigma}}\, e^{-i\pi/4}\right)$$

when $\frac{\epsilon_0 \omega}{\sigma} \ll 1$

$$\Rightarrow \quad \tilde{r} \approx -1 + 2\sqrt{\frac{\epsilon_0 \omega}{\sigma}}\, e^{-i\pi/4}$$

$$\tilde{r} = -1 + \sqrt{\frac{2\epsilon_0 \omega}{\sigma}} - i\sqrt{\frac{2\epsilon_0 \omega}{\sigma}}$$

$\Rightarrow$ Intensity reflection

$$\boxed{R = |\tilde{r}|^2 \approx 1 - 2\sqrt{\frac{2\omega\epsilon_0}{\sigma}}}$$

⇒ For a good conductor, almost all of the wave is reflected.

⇒ Almost a node in the wave at conductor surface

⇒ Tiny amount propagates in (skin depth)

⇒ ~~tiny phase shift~~; Reflected wave almost 180° out of phase with incident wave

⇒ Tiny extra phase shift for non-perfect conductor

Sketch of intensity